

A Priori Estimates for Two-Dimensional Water Waves with Angled Crests

Rafe H. Kinsey*

Sijue Wu†

Abstract

We consider the two-dimensional water wave problem in the case where the free interface of the fluid meets a vertical wall at a possibly non-right angle; our problem also covers interfaces with angled crests. We assume that the fluid is inviscid, incompressible, and irrotational, with no surface tension and with air density zero. We construct a low-regularity energy and prove a closed energy estimate for this problem. Our work differs from earlier work in that, in our case, only a degenerate Taylor stability criterion holds, with $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$, instead of the strong Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq c > 0$.

Contents

1	Introduction	3
1.1	Water wave problems	3
1.2	Outline of the paper	6
1.3	Notations, conventions and function spaces	6
2	The surface equation in the Lagrangian framework	8
2.1	The quasilinear equation	8
2.2	The degeneracy of the Taylor stability criterion	9
2.3	Boundary behavior and a special derivative	9
3	The Riemann mapping version	9
3.1	The Riemannian coordinates and notations	10
3.2	The water wave equations in the Riemann mapping framework	11
3.2.1	\mathcal{A} and the quantity A_1	12
3.2.2	Degenerate Taylor stability criterion and the singularity of the surface	13
3.2.3	\mathcal{A}_t	13
3.2.4	Formulas for $\frac{a_t}{a}$ and $\frac{\mathcal{A}_t}{\mathcal{A}}$	14
4	The main result	15
4.1	Definition of the energy	15
4.2	Discussion	16
4.3	The main result	16
4.4	The proof	17
4.4.1	The estimate for E_a	17
4.4.2	The estimate for E_b	19
4.4.3	The proof for $ z_{tt}(\alpha_0, t) - i $	19
4.5	Outline of the remainder of the proof	19

*R.H. Kinsey is supported in part by a University of Michigan Rackham Regents Fellowship and by NSF grants DMS-0800194 and DMS-1101434.

†S. Wu is supported in part by NSF grant DMS-1101434.

5	Quantities controlled by our energy	20
5.1	Basic quantities controlled by the energy	20
5.2	Controlling $\left\ \frac{a_t}{a} \right\ _{L^\infty}$	23
5.3	Controlling $\left\ \partial_{\alpha'} \frac{1}{\bar{z}_{,\alpha'}} \right\ _{L^2}$	23
5.4	Controlling $\left\ \frac{h_{t\alpha}}{h_\alpha} \right\ _{L^\infty}$	24
5.5	Controlling $\left\ \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\ _{L^\infty}$	25
5.6	Controlling $\left\ D_{\alpha'} \frac{1}{\bar{z}_{,\alpha'}} \right\ _{L^\infty}$ and $\left\ (\bar{z}_{tt} - i) \partial_{\alpha'} \frac{1}{\bar{z}_{,\alpha'}} \right\ _{L^\infty}$	25
5.7	Controlling $\left\ (I + \mathbb{H}) D_{\alpha'} Z_t \right\ _{L^\infty}$	26
5.8	Controlling $\left\ \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{\bar{z}_{,\alpha'}} \right\ _{L^\infty}$ and $\left\ (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{\bar{z}_{,\alpha'}} \right\} \right\ _{L^\infty}$	26
6	Controlling $\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t \theta ^2 h_\alpha d\alpha$	27
7	Controlling $\Re \int i \left(\frac{h_\alpha^2}{ z_\alpha ^2} \right)_t \theta_\alpha \bar{\theta} d\alpha$	29
8	Controlling G_θ of E_b	33
9	Controlling G_θ of E_a	35
10	A characterization of the energy	46
10.1	A characterization of the energy in terms of position and velocity	46
10.1.1	The proof	47
10.1.2	Controlling $\left\ \bar{z}_{tt,\alpha'} \right\ _{L^2}$	48
10.1.3	Controlling $\left\ D_{\alpha'}^2 \bar{z}_{tt} \right\ _{L^2}$	48
10.2	Singularities and the angle of the crest	52
	Appendices	53
A	Holomorphicity, mean and boundary behavior	53
A.1	The Hilbert transform \mathbb{H}	53
A.2	Boundary properties	54
A.3	Holomorphic functions and what disappears under $(I - \mathbb{H})$	56
A.3.1	Identities	57
A.3.2	Mean conditions	58
B	Useful inequalities and identities	58
B.1	Sobolev inequalities and the Peter-Paul trick	59
B.2	Derivatives and complex-valued functions	59
B.3	Hardy's inequality and commutator estimates	60
B.4	The $\dot{H}^{1/2}$ norm	62
B.5	Commutator identities	62
C	Summary of notation	63
D	Main quantities controlled	64
	References	64

1 Introduction

1.1 Water wave problems

A class of water wave problems concerns the dynamics of the free surface separating a zero-density region (air) from an incompressible fluid (water), under the influence of gravity.

Let $\Omega(t)$ be the fluid region, $\Sigma(t)$ be the free surface between the fluid and the air, and Υ , if it exists, be the fixed rigid boundary of $\Omega(t)$ (e.g., the ocean bottom or the coast), for time $t \geq 0$; thus $\partial(\Omega(t)) = \Sigma(t) \cup \Upsilon$. We will henceforth assume that the fluid is not only incompressible but also irrotational, and we will neglect surface tension and viscosity. Assume that the fluid density is 1. If the gravity field is $-\mathbf{j}$, the governing equations of motion are

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{j} - \nabla P \quad \text{on } \Omega(t) \quad (1)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{on } \Omega(t) \quad (2)$$

$$\operatorname{curl} \mathbf{v} = 0 \quad \text{on } \Omega(t) \quad (3)$$

$$P = 0 \quad \text{on } \Sigma(t) \quad (4)$$

$$(1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)) \quad (5)$$

$$\mathbf{v} \text{ is tangent to the fixed boundary } \Upsilon \text{ (if it exists),} \quad (6)$$

where \mathbf{v} is the fluid velocity and P is the fluid pressure. We also assume that

$$\mathbf{v}(x, y, t) \rightarrow 0 \text{ as } y \rightarrow -\infty. \quad (7)$$

An important quantity governing the stability of these problems is $-\frac{\partial P}{\partial \mathbf{n}}$, where \mathbf{n} is the outward-facing unit normal vector to $\Sigma(t)$.

The study of water waves dates back centuries; early mathematical works include those by Stokes [Sto47], Levi-Civita [LC25] and G.I. Taylor [Tay50]. [Nal74], [Yos82], and [Cra85] obtained early local-in-time existence results for the two-dimensional water wave problem for small data. In 1997, [Wu97] proved, for the infinite depth two-dimensional water wave problem (1)-(5) and (7) with $\Upsilon = \emptyset$, that the Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq c > 0$ always holds and that the problem is locally well-posed in Sobolev spaces H^s , $s > 4$, for arbitrary data. [Wu99] proved a similar result in three dimensions. Since then, there have been numerous results on local well-posedness in both two and three dimensions, for the water wave equations with nonzero vorticity [CL00] [Lin05] [CS07], with a fixed bottom [Lan05], and with nonzero surface tension [AM05] [CS07]. [ABZar] proved local well-posedness of the problem for interfaces in $C^{3/2+\varepsilon}$.¹ All of these works assume the strong Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq c > 0$. In addition, in the past several years [Wu09], [Wu11], [GMS12], [IP13], [AD13] and [HIT14] have proved results showing almost global or global well-posedness in two and three dimensions for sufficiently small and smooth initial data.

All of these results assume either no fixed boundary or else a fixed boundary that is a positive distance away from the free interface. In actual oceans, of course, the free surface does intersect with the rigid boundary, e.g., on the coast. It is important, therefore, to study the water wave problem in this setting. In this paper, we will begin such a program, by considering the two-dimensional water wave equations in the presence of a rigid vertical wall that interacts with the free interface.

In addition to addressing the interaction of the free surface with the rigid vertical wall, our research covers singularities in water waves away from the rigid boundary. Indeed, by Schwarz reflection, a model of waves making a non-right angle with a rigid vertical wall corresponds to a symmetric angled crest in the middle of the ocean; see Figure 1. More generally, our work applies to water waves with very low regularity, including those with angled crests.² Our work differs from the low-regularity result of [ABZar], which assumes that the strong Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq c > 0$ holds and considers surfaces in $C^{3/2+\varepsilon}$. In our case, we allow

¹A recent preprint, [ABZ14b], improves on these results using Strichartz estimates, reducing the required smoothness of the interface to a fraction under $C^{3/2}$. [HIT14] also has a low-regularity result that improves on [ABZar] for 2-D water waves. Both [ABZ14b] and [HIT14] work in the setting where the strong Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq c > 0$ holds.

²We note that the angled crests here don't have to be symmetric.

surfaces that are not even in C^1 where only a degenerate Taylor stability criterion $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$ holds.³ Our research relates to recent work of Wu [Wu12], who constructed a class of convection-dominated self-similar solutions of the two-dimensional water wave equation with such angled crests. Our work indicates that the water wave equations (1)-(7) admit solutions with this type of singularity.

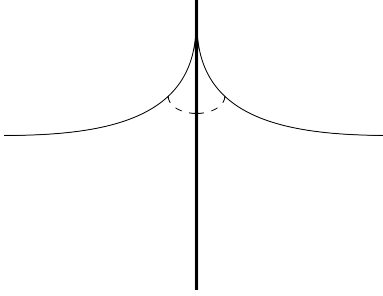


Figure 1: Under a Schwarz reflection, a non-right angle at a vertical wall corresponds to a symmetric angled crest in the middle of surface.

For simplicity, we consider a rigid boundary consisting of two vertical walls, with water of infinite depth in between the walls. Working in two dimensions $(x, y) \in \mathbb{R}^2$, we assume that our fixed walls are at $x = 0, 1$, with the fluid region $\Omega_0(t) \subset [0, 1] \times (-\infty, c)$ for some $c < \infty$. We assume that the water surface makes a right angle with the wall $x = 0$, but we allow a non-right (i.e., not 90°) angle at $x = 1$.^{4 5}

This angle will be of fundamental importance for the remainder of the paper.⁶ We will use ν to denote this angle; see Figure 2. If there is a right angle at $x = 1$, then existing techniques ought to apply, so long as there are no singularities in the middle of the free surface. What's novel about our work is that it extends to the non-right angle regime, and allows singularities in the middle of the free surface.

We begin by symmetrizing the problem, using Schwarz reflection to expand the domain across the y -axis, using the fact that (6) implies that $v_1(0, y, t) \equiv 0$. Precisely, for $\mathbf{v} = (v_1, v_2)$ and $x \in [0, 1]$, we define

$$\mathbf{v}(-x, y, t) = (-v_1(x, y, t), v_2(x, y, t)). \quad (8)$$

It is easy to check that equations (1) through (7) continue to hold in the expanded domain. Henceforth, we shall work exclusively in this reflected domain $\Omega(t)$. $\Omega(t) \subset [-1, 1] \times (-\infty, c)$ is symmetric with respect to the y -axis, with solid vertical walls Υ at $x = -1, 1$. When we say the *wall* we will mean $x \equiv \pm 1$, and when we say the *corners*, we will mean the corner of the water surface $\Sigma(t)$ at $x = \pm 1$. See Figure 2.

³In [ABZar], Alazard et al. consider the water wave problem in the presence of a bottom bounded away from the interface. In this framework, the strong Taylor stability condition doesn't necessarily hold, so it is assumed. In our case, the only rigid boundary is the vertical wall and we can prove that the degenerate Taylor stability condition always holds. The degeneracy occurs at the singularity in the interface and when the water wave meets the wall with a non-right angle.

⁴Although in our problem there is a vacuum-liquid-solid triple point, one should not confuse it with the dynamic contact angle problem, where typically surface tension is very important: our problem and the contact angle problem are two different problems.

⁵We started studying our problem by focusing only on resolving the difficulties when the free surface makes a non-right angle ν at the wall $x = 1$. The result we obtain, however, covers cases where the free surface has angled crests; in particular, the angle of the water wave at $x = 0$ doesn't have to be 90° .

⁶We will see in §3.2.2 that the angle ν must be no more than $\frac{\pi}{2}$. Therefore a non-right angle ν is necessarily $< 90^\circ$.

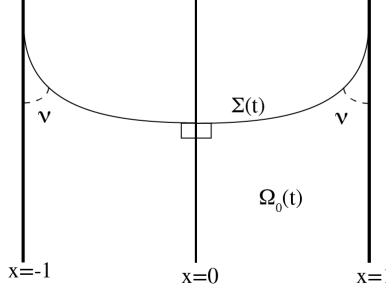


Figure 2: The fluid domain $\Omega_0(t)$ has solid walls at $x = 0, 1$, with a possibly non-right angle ν at $x = 1$ and a right angle at $x = 0$. The figure shows the reflected domain $\Omega(t)$ under a Schwarz reflection across $x = 0$, with a symmetric free surface $\Sigma(t)$.

Allowing non-right angles at the wall and singularities in the surface introduces significant challenges.

In [Wu97], a key result was proving that the Taylor sign condition $-\frac{\partial P}{\partial \mathbf{n}} \geq c > 0$ always holds as long as the surface is smooth and non-self-intersecting. In the situation where $-\frac{\partial P}{\partial \mathbf{n}} < 0$, [Ebi87] has shown that the free boundary problem is not well-posed, a fact which is not surprising, since it is the positive sign of $-\frac{\partial P}{\partial \mathbf{n}}$ that gives the hyperbolicity of the equation. In our situation, however, we will have that $-\frac{\partial P}{\partial \mathbf{n}} = 0$ at the wall when there is a non-right angle, and at the points on the surface where there are singularities (e.g., angled crests).⁷ In this case, the water wave equation is degenerate hyperbolic, and it is harder to construct an energy that can be closed.

A second challenge comes from the potential asymmetry of certain quantities at the left and right boundaries of the domain. Thanks to the Schwarz reflection, we can now treat our problem as being periodic. The possibly non-right angle of the free surface with the wall introduces a fundamental asymmetry: if the surface is sloping downwards near $x = -1$, it is sloping upwards near $x = 1$. Therefore, the values of certain quantities at $x = -1$ will not agree with their values at $x = 1$. This leads to problems both with integration by parts and with handling estimates of various commutators involving the (periodic) Hilbert transform. This challenge actually proves to be a blessing in disguise, because it directs us to precisely the quantities that *are* well-behaved as periodic functions.

A third challenge also proves to have a silver lining. [Wu97] relied on the Riemann mapping to flatten out the fluid domain. When the angle of the free surface with the wall is not 90° , or when there is an angled crest in the middle of the free surface, the Riemann mapping has a singularity. Instead of avoiding the Riemann mapping, we embrace it, since it directly captures the geometry of the interface, and gives us heuristics to determine which quantities we might or might not be able to control. We used these heuristics, and more importantly the self-similar solution constructed in [Wu12], to guide us in the construction of our energy.

In this paper, we prove an a priori estimate for solutions of the water wave equation in this framework. We follow the general approach of Wu's earlier papers [Wu97] and [Wu99], in reducing the water wave problem to an equation on the free surface, differentiating the equation with respect to time, and then using the fact that the velocity is antiholomorphic to express the nonlinearities in terms of commutators involving the Hilbert transform. The novelty is that we introduce a new energy that relies on two different singular weights and controls some natural holomorphic quantities, and that our estimates *do not* depend on any positive lower bound for $-\frac{\partial P}{\partial \mathbf{n}}$.⁸ Substantial technical difficulties exist due to the very low regularity involved in the energy and the lack of a positive lower bound for $-\frac{\partial P}{\partial \mathbf{n}}$.⁹ Nevertheless, we have overcome these challenges.

⁷Precisely, $\nabla P = 0$ at the wall when $\nu \neq \frac{\pi}{2}$ and at the points on the free surface where there are angled crests.

⁸We will show that in our framework $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$. In fact, a positive lower bound on $-\frac{\partial P}{\partial \mathbf{n}}$ is roughly equivalent to a non-singular Riemann mapping or, equivalently, a non-singular surface; see §3.2.2.

⁹In particular, we were led to an energy consisting only of a limited and specific collection of weighted quantities rather than the usual assortment of quantities in more traditional energies (e.g., squares of classical Sobolev space norms). See the characterization of the energy at (293) and (294) in §10.1.

Our energy inequality is a crucial step towards proving local existence in this framework. This will be the focus of an upcoming paper.

We mention that our energy is finite for the self-similar water waves constructed in [Wu12].

1.2 Outline of the paper

In the next subsection, §1.3, we present some of the notations and conventions and introduce the function spaces and norms to be used in the paper. Then, in §2, we introduce the Lagrangian framework for the water wave problem, following [Wu97] and [Wu09]. In particular, in §2.3, we discuss some of the boundary challenges mentioned above, and introduce a special derivative that we will use in our energies. In §3, we introduce the Riemann mapping and use it to derive a form of the equation in “Riemannian coordinates.” Finally, in §4, we define the energy (in §4.1) and state our main result, the a priori inequality (in §4.3). We offer some discussion on our choice of energy in §4.2. We begin the proof in §4.4, and then in §4.5 we outline the remainder of the proof, which takes up sections §5 through §9.

In §10 we give a characterization of the energy in terms of the velocity and position of the free surface, and we discuss the types of singularities allowed when our energy is finite.

The derivation of the free surface equations and the proof of the main result rely on understanding the boundary behaviors, the holomorphicity, and the means of various quantities; we leave these, as well as some basic identities and inequalities used in the proof of the main result, to appendices §A–§B. The reader may want to read these appendices before certain sections in the main text. We have two additional appendices that might be useful to the reader. In §C, we provide an overview of the notation used in the paper, with cross references to where everything was initially defined. In §D, we list various quantities controlled by the energy, again with cross references.

1.3 Notations, conventions and function spaces

We will define most of our notations throughout the text, as we introduce our various quantities. Here we only list some general conventions and notations.

Since we are in two dimensions, we will often work in complex coordinates $(x, y) = x + iy$. We will use $\Re z := x$ and $\Im z := y$ to represent the real and imaginary parts, respectively, of $z = x + iy$.

Compositions are always in terms of the spatial variable. For example, for $f = f(\alpha, t)$, $g = g(\alpha, t)$, we define $f \circ g = f \circ g(\alpha, t) := f(g(\alpha, t), t)$. An expression $f_x(x, y)$ means $\partial_x f(x, y)$; we occasionally use the notation f' , which is always the spatial derivative in whatever coordinates we are using.

Once we have reduced the water wave equations to an equation in one spatial dimension, we will primarily be working with the spatial domain $I := [-1, 1]$. We will often refer to the “boundary”; this refers to what happens at ± 1 . We write $f|_{\partial} := f(1) - f(-1)$. We will use

$$\oint_I f := \frac{1}{|I|} \int_I f(x) dx = \frac{1}{2} \int_{-1}^1 f(x) dx \quad (9)$$

for the mean of a function f . Here, and elsewhere for other integrals, when there is no risk of ambiguity, we will often drop the subscript I .

We will often use commutators. We define

$$[A, B] := AB - BA. \quad (10)$$

We will use the following notation as an abbreviation for a type of higher-order Calderon commutator:

$$[f, g; h](\alpha') := \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta'. \quad (11)$$

We will use C as a placeholder to refer to a universal constant, possibly varying from line to line. We will also often use the notation $f \lesssim g$, which means that there exists some universal constant C such that $f \leq Cg$.

We will at several points have long series of identities or inequalities. When we say “on the RHS” of an equation block with a string of multiple equalities or inequalities, we mean all the terms on the right hand side of the last equality or inequality sign in the string. Similarly, when we say “on the LHS,” we mean all the terms to the left of the very first equality or inequality sign in the string of equalities and inequalities. We have tried to avoid saying “on the n th line” when any of the mathematical formulas splits into more than one typographic line, but if we have, “line” refers to the mathematical, not typographic, line.

We have tried to give extensive cross references for each time we use a result or estimate. We tend to refer to equation numbers, rather than propositions, since it seems that these will be easier to find as cross references. When we refer to an equation number as part of a proposition, we are of course referring to the whole proposition, including any conditions assumed.

When we are deriving estimates, we sometimes use the cross references within our equations, e.g.:

$$\begin{aligned} f &\leqslant g \\ &\leqslant h \end{aligned} \tag{12}$$

and

$$\begin{aligned} h &\lesssim j + f \\ &\lesssim j + (12) \\ &\lesssim j + h. \end{aligned} \tag{13}$$

This means (12) is used to obtain (13). We hope this will help the reader locate the previous estimate or estimates.

In several of our more complicated estimates, we will split terms up $f = I + II$ and then $I = I_1 + I_2$, $I_1 = I_{11} + I_{12}$, etc. Such notation will be *local to each section*. There is an ambiguity between the use of I as a placeholder, its use as the identity operator, and its use as $I := [-1, 1]$. It should be clear from the context which one is being used.

We now introduce the function spaces and norms we will use. We will work with functions $f(\cdot, t)$ defined on $I = [-1, 1]$. Except when necessary to avoid ambiguity, we neglect to write the time variable; when it is not specified, function spaces and norms are in terms of the spatial variable.

We say $f \in C^k(J)$, $J = (-1, 1)$ or $[-1, 1]$, if for every $0 \leqslant l \leqslant k$, $\partial_x^l f$ is a continuous function on the interval J . We say $f \in C^k(S^1)$ (i.e., periodic C^k) if for every $0 \leqslant l \leqslant k$, $\partial_x^l f \in C^0[-1, 1]$ and $\partial_x^l f(1) = \partial_x^l f(-1)$. ($\partial_x^l f$ at the endpoints 1 or -1 is the derivative from the left or right, respectively.) Note in particular that saying $f \in C^0(S^1)$ implies that $f|_{\partial} = 0$.

For $1 \leqslant p < \infty$, we define our L^p spaces by the norms

$$\|f\|_{L^p} := \|f\|_{L^p(I)} := \left(\int_{[-1, 1]} |f|^p \right)^{1/p}, \tag{14}$$

and we define L^∞ analogously. We will sometimes deal with weighted L^p spaces. We write

$$\|f\|_{L^p(\omega)} = \|f\|_{L^p(\omega dx)} := \left(\int_I |f(x)|^p \omega(x) dx \right)^{1/p} \tag{15}$$

for weights $\omega \geqslant 0$. Whenever we write L^p , we will be referring to $L^p(I)$, in the spatial variable. For weighted L^p spaces, we always write $L^p(\omega)$ or $L^p(\omega dx)$, where ω is the weight function.

Note that if $f \in C^0(S^1)$, with $f' \in L^p$, then if we extend f periodically to $(-3, 3)$, the weak derivative of the extended function is the periodic extension of f' to $(-3, 3)$ and is in $L^p(-3, 3)$.

We define the homogeneous half-derivative space $\dot{H}^{1/2}$ by the norm

$$\|f\|_{\dot{H}^{1/2}} := \left(\frac{\pi}{8} \iint_{I \times I} \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\alpha' d\beta' \right)^{1/2}. \tag{16}$$

Through the remainder of the paper, when we say the boundary value of a function G defined on the fluid region $\Omega(t)$ (resp., on $P^- := I \times (-\infty, 0]$), we mean the value of G on the free surface (resp., on $I \times \{0\}$); we do not include the value on vertical walls $x = \pm 1$. Except when there's a risk of confusion, we will slightly abuse notation and say that a function f on the free surface (resp., on $I \times \{0\}$) is “holomorphic” (or “antiholomorphic”); what we mean, precisely, is that it is the boundary value of a function that is holomorphic (or antiholomorphic) in the fluid region $\Omega(t)$ (resp., on P^-).

2 The surface equation in the Lagrangian framework

Let $z(\alpha, t) = x(\alpha, t) + iy(\alpha, t)$, $\alpha \in I = [-1, 1]$ be a parametrization of the free surface $\Sigma(t)$ in the *Lagrangian* variable α , i.e., $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity and z_{tt} is the acceleration of the particle occupying position $z(\alpha, t)$ at time t . We choose the initial parametrization to be in arc-length coordinates: $|z_\alpha(\alpha, 0)| \equiv 1$.¹⁰ Along the free surface, the Euler equation (1) is $z_{tt} + i = -\nabla P$. By equation (4), we know ∇P is orthogonal to the free surface. Since iz_α is normal to the free surface, we can rewrite our main equation as

$$z_{tt} + i = i\mathbf{a}z_\alpha, \quad (17)$$

where

$$\mathbf{a} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|} \in \mathbb{R} \quad (18)$$

for $\frac{\partial P}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla P$ the outward-facing normal derivative. The incompressibility and irrotationality conditions (2) and (3) imply that the conjugate velocity is holomorphic; therefore \bar{z}_t is the boundary value of a holomorphic function in the fluid region.

2.1 The quasilinear equation

We henceforth focus on the equations on the free surface.¹¹ As in [Wu97] and following works, we differentiate (17) with respect to time and take conjugates, turning it into the quasilinear equation¹²

$$\bar{z}_{ttt} + i\mathbf{a}\bar{z}_{t\alpha} = -i\mathbf{a}_t\bar{z}_\alpha, \quad (19)$$

where we continue to have \bar{z}_t the boundary value of a holomorphic function. This is the basic equation we will work with throughout the paper.

The holomorphicity of \bar{z}_t implies that $\mathbf{a}\partial_\alpha\bar{z}_t$ can be written as $\mathbf{a}\partial_\alpha\mathfrak{H}\bar{z}_t$, where \mathfrak{H} is the Hilbert transform with respect to the free surface $\Sigma(t)$. Because $i\partial_\alpha\mathfrak{H}$ is a nonnegative operator, a natural positive energy can be defined.

In [Wu99] and [Wu09], coordinate-independent formulas for the RHS were derived, using the holomorphicity of \bar{z}_t and the invertibility of the double-layer potential. We will instead follow the original approach of [Wu97], relying on the Riemann mapping version of the equation to derive the RHS. We do so in §3.2.3.

¹⁰We choose this initial parametrization for the sake of convenience in discussion. Our a priori estimate is a purely geometric one and does not depend on this parametrization. On the other hand, if $|z_\alpha(\cdot, 0)| \equiv 1$, then $|z_\alpha(\cdot, t)|$ is bounded away from 0 and infinity so long as our energy $E(t)$ remains finite. This follows from $\partial_t|z_\alpha| \leq |z_\alpha||D_\alpha z_t|$, $\partial_t|1/z_\alpha| \leq |1/z_\alpha||D_\alpha z_t|$, $\|D_\alpha z_t\|_{L^\infty} \leq C(E(t))$ (see §5.1), and the Gronwall inequality.

¹¹We may solve for the velocity on $\Omega(t)$ from its boundary values (including the condition that it goes to zero as $y \rightarrow -\infty$), and then solve for the pressure from the velocity.

¹²We call it “quasilinear” because in the classical situation [Wu97], this equation is quasilinear with the RHS the lower-order term. However, in our setting, due to the degeneracy of $-\frac{\partial P}{\partial \mathbf{n}}$ we do not know a priori that this is still the case; only by our proof do we show that the RHS is, indeed, lower-order and that (19) is in fact quasilinear. All references to (19) and related equations being “quasilinear” should be interpreted with this in mind.

2.2 The degeneracy of the Taylor stability criterion

Recall that in [Wu97] it was essential to show the *strong Taylor stability criterion* $-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$. Here we are not so lucky, since we only have $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$. We will prove this fully in §3.2.1 and §3.2.2. For now, we make a quick observation: ∇P is degenerate when the free surface makes a non-right angle at the corner.¹³ Indeed, from (17), we have

$$\frac{x_\alpha}{-y_\alpha} = \frac{y_{tt} + 1}{x_{tt}}. \quad (20)$$

Along the wall, we know that $x_t \equiv 0$ by (6), so we have $x_{tt} \equiv 0$. Observe that the angle ν between the water surface and the wall $x = -1$ is given by

$$\tan \nu = \frac{x_\alpha}{-y_\alpha}. \quad (21)$$

If $(y_{tt} + 1)/x_{tt}$ is infinite, then the angle must be 90° . Therefore, the only possible way for the angle of the free surface with the wall to be other than a right angle is if the numerator $y_{tt} + 1$ is zero at the corner. This implies that $\nabla P = 0$ at the corner, hence $-\frac{\partial P}{\partial \mathbf{n}} := -\mathbf{n} \cdot \nabla P = 0$ at the corner.

2.3 Boundary behavior and a special derivative

We discuss here the boundary behavior of our basic quantity \bar{z}_t and define a weighted spatial derivative that will be used in constructing higher-order energies. For this paper we will also need to understand the boundary properties of some other quantities. We will leave those to §A.2 in the appendices.

In our framework, the conjugate velocity $\bar{\mathbf{v}}$ has real part $\Re \bar{\mathbf{v}}$ odd and imaginary part $\Im \bar{\mathbf{v}}$ even, and $\Re \mathbf{v}(\pm 1, y) = 0$ by (6). Therefore, $\bar{\mathbf{v}}$ is a periodic holomorphic function in $\Omega(t)$, with $\bar{\mathbf{v}}(-1, y) = \bar{\mathbf{v}}(1, y)$, and our basic function \bar{z}_t satisfies

$$\bar{z}_t|_\partial = 0. \quad (22)$$

We would like the quantities we work with to have such periodic boundary behavior, since we often integrate by parts and since periodicity will be essential for certain results involving the periodic Hilbert transform that we will use. Periodicity is not preserved by the spatial derivative ∂_α ; for example, we do not necessarily have $\partial_\alpha \bar{z}_t|_\partial = 0$. Notice, however, that for holomorphic functions, $\partial_z = -i\partial_y$ and ∂_y preserves periodicity. We therefore look for the derivative on the surface $\Sigma(t)$ that corresponds to ∂_z inside the domain. This derivative is

$$D_\alpha := \frac{1}{z_\alpha} \partial_\alpha. \quad (23)$$

Indeed, if $g(\alpha, t) = G(z(\alpha, t), t)$, and G is holomorphic, then $\partial_\alpha g = (G_z \circ z) z_\alpha$ so $D_\alpha g = (\partial_z G) \circ z = -i(\partial_y G) \circ z$. Thus $D_\alpha^k g$ is the boundary value of holomorphic function $\partial_z^k G$, provided G is holomorphic. $D_\alpha^k g$ is in addition periodic for any $k \geq 1$, so long as G is periodic and holomorphic.

We may therefore conclude that $D_\alpha^k \bar{z}_t$ is the boundary value of holomorphic function $\partial_z^k \bar{\mathbf{v}}$ in $\Omega(t)$, and

$$D_\alpha^k \bar{z}_t|_\partial = 0, \text{ for any } k \geq 0. \quad (24)$$

We will use D_α as the spatial derivative in constructing higher-order energies. In addition to preserving holomorphicity and periodic boundary behavior, it transforms well under the Riemann mapping, to be discussed in the next section.

3 The Riemann mapping version

We now introduce a version of water wave equations using the Riemann mapping to flatten out the curved free interface.¹⁴ The Riemann mapping version of the equations offers a key advantage. Because we are

¹³This observation has also been made by [ABZ14a] in §6.1.

¹⁴The idea of using Riemann mapping to study the wellposedness of 2-d water waves dates back to [Wu97].

working on a flat domain, our Hilbert transform is now the periodic Hilbert transform \mathbb{H} defined by

$$\mathbb{H}f(\alpha') := \frac{1}{2i} \text{pv} \int_I \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta') d\beta'. \quad (25)$$

Since $\mathbb{H}f \in i\mathbb{R}$ for f real-valued, this allows us to invert the operator $(I - \mathbb{H})$ on purely real (resp., purely imaginary) functions by taking real (resp., imaginary) parts.

The problem with using the Riemann mapping in our framework is that the possibly non-right angle at the corner and singularities in the middle of the interface can create singularities in the Riemann mapping. This differs from previous work (e.g., [Wu97]), where there were no such singularities. On the other hand, the Riemann mapping is helpful for us in that it gives us heuristic information about the nature of our singularities, which leads us to the right energy functional.

3.1 The Riemannian coordinates and notations

Let $\Omega_0(t) \subset [0, 1] \times (-\infty, c)$ be the unreflected fluid domain, and let

$$\Phi_0 : \Omega_0(t) \rightarrow P_0^- := \{(x, y) : x \in [0, 1], y \leq 0\} \quad (26)$$

be the unique Riemann mapping that takes the two upper corners to $(0, 0)$ and $(1, 0)$ and ∞ to ∞ . Let Φ be the Schwarz reflection of Φ_0 ,

$$\Phi : \Omega(t) \rightarrow P^- := \{(x, y) : x \in [-1, 1], y \leq 0\}. \quad (27)$$

We know Φ is a biholomorphic map that takes the free surface $\Sigma(t)$ to $I \times \{0\}$, and wall to wall. Let

$$\alpha' = h(\alpha, t) := \Phi(z(\alpha, t), t) : I \rightarrow I \quad (28)$$

be the change of coordinates taking the Lagrangian variable α to the Riemannian variable α' , and let h^{-1} be the spatial inverse of h , defined by $h(h^{-1}(\alpha', t), t) = \alpha'$. We define

$$Z(\alpha', t) := z \circ h^{-1}(\alpha', t) = z(h^{-1}(\alpha', t), t). \quad (29)$$

$Z = Z(\alpha', t)$ is a parametrization of the free surface $\Sigma(t)$ in Riemannian variable α' . We write

$$\begin{aligned} Z_t &:= z_t \circ h^{-1}; & Z_{tt} &:= z_{tt} \circ h^{-1}; \\ Z_{,\alpha'} &:= \partial_{\alpha'} Z; & Z_{t,\alpha'} &:= \partial_{\alpha'} Z_t; & Z_{tt,\alpha'} &:= \partial_{\alpha'} Z_{tt}; \text{ etc.} \end{aligned} \quad (30)$$

and

$$\mathcal{A} := (\mathfrak{a}h_\alpha) \circ h^{-1}; \quad \mathcal{A}_t := (\mathfrak{a}_t h_\alpha) \circ h^{-1}. \quad (31)$$

Observe that $Z = z \circ h^{-1} = \Phi^{-1}$. Therefore

$$Z_{,\alpha'}(\alpha', t) = \partial_{\alpha'}(\Phi^{-1}(\alpha', t)), \quad \text{and} \quad \frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z. \quad (32)$$

Observe also that under the change of variables $(D_\alpha f) \circ h^{-1} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}(f \circ h^{-1})$. We therefore define

$$D_{\alpha'} := \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}. \quad (33)$$

We note that for holomorphic functions on P^- , $\partial_{\alpha'} = \partial_{z'}$, so $\partial_{\alpha'}$ also preserves holomorphicity. However, it has two drawbacks: it does not preserve periodicity and, for non-right angles or singular surfaces, it is more singular than $D_{\alpha'}$. We will give more details in §3.2.2 after deriving some necessary formulas.

3.2 The water wave equations in the Riemann mapping framework

We now derive the water wave equations in Riemann mapping framework. We follow the approach of [Wu97], although we work with the real and imaginary parts together instead of separating $Z_t = X_t + iY_t$ into real and imaginary parts.

In this section, we will present a full derivation of the RHS of the quasilinear equation in Riemannian coordinates. This won't end up be the precise way we estimate these quantities for higher-order energies; instead, we will have to do those by hand, in later sections.¹⁵ Rather, the derivation in this section proves useful because it provides several key formulas, especially one that allows us to estimate the crucial quantity $\frac{a_t}{a}$. Still, it seems most natural to motivate the following calculations by writing as if our goal were to derive the full quasilinear Riemann mapping version of the equations, rather than just to get a technical formula for $\frac{a_t}{a}$.

Beginning with the conjugated form of our equation (17) and with (19), we precompose both sides with h^{-1} to get the surface equations in the flattened Riemann mapping coordinate:

$$\overline{Z}_{tt} - i = -i\mathcal{A}\overline{Z}_{,\alpha'}; \quad (34)$$

$$\overline{Z}_{ttt} + i\mathcal{A}\overline{Z}_{t,\alpha'} = -i\mathcal{A}_t\overline{Z}_{,\alpha'}, \quad (35)$$

where \overline{Z}_t is the boundary value of the periodic holomorphic function $\overline{\mathbf{v}} \circ \Phi^{-1}$, with $\overline{Z}_t|_{\partial} = 0$.

The following proposition gives a characterization of the boundary value of a periodic holomorphic function on P^- .

Proposition 1. *a. Let $g \in L^p$ for some $p > 1$. Then g is the boundary value of a holomorphic function G on P^- satisfying $G(-1 + iy) = G(1 + iy)$ for all $y < 0$ and $G(x + iy) \rightarrow c_0$ as $y \rightarrow -\infty$ if and only if*

$$(I - \mathbb{H})g = c_0. \quad (36)$$

Moreover, $c_0 = \int_I g$.

b. Let $f \in L^p$ for some $p > 1$. Then $\mathbb{P}_H f := \frac{1}{2}(I + \mathbb{H})f$ is the boundary value of a periodic holomorphic function \mathcal{F} on P^- , with $\mathcal{F}(x + iy) \rightarrow \frac{1}{2}\int_I f$ as $y \rightarrow -\infty$.

Proposition 1 is a classical result, which can be proved by the Cauchy integral formula; see [Jou83]. We note that our basic quantity $D_{\alpha'}^k \overline{Z}_t$ is the boundary value of the periodic holomorphic function $\partial_z^k (\overline{\mathbf{v}} \circ \Phi^{-1})$ and satisfies

$$(I - \mathbb{H})D_{\alpha'}^k \overline{Z}_t = 0, \quad D_{\alpha'}^k \overline{Z}_t|_{\partial} = 0; \quad \text{for } k \geq 0 \quad (37)$$

by (369) and (347); see discussions therein.

We define the following projection operators:

$$\mathbb{P}_H f := \frac{(I + \mathbb{H})}{2} f; \quad \mathbb{P}_A f := \frac{(I - \mathbb{H})}{2} f. \quad (38)$$

We will refer to \mathbb{P}_H as the “holomorphic projection” and \mathbb{P}_A as the “antiholomorphic projection.” These operators are, indeed, proper projections when interpreted modulo a constant.

We now seek formulas for \mathcal{A} and \mathcal{A}_t . For derivations here and hereafter, we will rely on the technicalities given in appendices §A-§B. We will work under the assumption that all the regularity properties required in appendices §A-§B are met by our solutions.¹⁶

¹⁵Because we are using weighted derivatives $D_{\alpha'}$ (to ensure periodic boundary behavior and prevent singularity), and because our energy involves only a selected collection of weighted quantities, we have to take more care in applying derivatives to the RHS than [Wu97] does. Another source of difficulty is that our derivative $D_{\alpha'}$ is not purely real, so inverting $(I - \mathbb{H})$ is more subtle.

¹⁶Note that a key step in the proof of an existence result is having a priori estimates for sufficiently regular solutions.

3.2.1 \mathcal{A} and the quantity A_1

We first derive a formula for \mathcal{A} . Historically, in [Wu97], this derivation showed that the strong Taylor stability criterion would automatically hold. There is now [Wu99] a direct proof of this via basic elliptic theory without using the Riemann mapping. For our purposes, though, this original derivation here will be crucial because it introduces a quantity, A_1 , that compares the degeneracy of $-\frac{\partial P}{\partial \mathbf{n}}$ directly with that of the Riemann mapping and therefore the geometry of the surface.

We begin with (34). The key observation is that $\bar{Z}_{tt} - i$ is in some sense close to holomorphic, since \bar{Z}_t is holomorphic. Therefore, if we apply $(I - \mathbb{H})$ to this, it should be “small” in some sense. If the RHS were purely imaginary, we would be able to invert $(I - \mathbb{H})$ by taking imaginary parts, and thus get a formula for the RHS. The RHS is not purely real or imaginary, but if we multiply both sides by the holomorphic $Z_{,\alpha'}$, it is:

$$Z_{,\alpha'}(\bar{Z}_{tt} - i) = -i\mathcal{A}|Z_{,\alpha'}|^2. \quad (39)$$

We will apply $(I - \mathbb{H})$ to both sides of (39). Before we do so, we must expand out \bar{Z}_{tt} . Let

$$F(z(\alpha, t), t) := \bar{z}_t(\alpha, t). \quad (40)$$

Note that $F = \bar{\mathbf{v}}$, a periodic holomorphic function in $\Omega(t)$. We will use this expansion several times in the sequel, always with this definition of F . By the chain rule,

$$\bar{z}_{tt} = \frac{d}{dt}F(z(\alpha, t), t) = (F_z \circ z)z_t + (F_t \circ z). \quad (41)$$

Recall from §2.3 that $\partial_z = D_\alpha$ for holomorphic functions. Therefore, $F_z \circ z = \frac{\bar{z}_{t\alpha}}{z_\alpha}$, and thus $\bar{z}_{tt} = \frac{\bar{z}_{t\alpha}}{z_\alpha}z_t + F_t \circ z$. We precompose with h^{-1} :

$$\bar{Z}_{tt} = \left(\frac{\bar{Z}_{t,\alpha'}}{Z_{,\alpha'}} \right) Z_t + F_t \circ Z. \quad (42)$$

We can now write our equation (39) as

$$\bar{Z}_{t,\alpha'}Z_t + Z_{,\alpha'}(F_t \circ Z) - iZ_{,\alpha'} = -i\mathcal{A}|Z_{,\alpha'}|^2. \quad (43)$$

We apply $(I - \mathbb{H})$ to both sides. By (370), (372) and (362), writing $(I - \mathbb{H})(Z_t \bar{Z}_{t,\alpha'}) = [Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'}$, we get

$$[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} - i = (I - \mathbb{H}) \left(-i\mathcal{A}|Z_{,\alpha'}|^2 \right). \quad (44)$$

We now take imaginary parts of both sides. This gives us the new quantity A_1 :

$$A_1 := \mathcal{A}|Z_{,\alpha'}|^2 = \Im \left(-[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} \right) + 1. \quad (45)$$

This is the same A_1 as that in [Wu97]. It's easy to see that $\Im \left(-[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} \right)$ is non-negative, by integration by parts. Indeed, if $Z_t = X_t + iY_t$, then

$$\begin{aligned} \Im \left(-[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'} \right) &= -\Im \frac{1}{2i} \int (Z_t(\alpha') - Z_t(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \bar{Z}_{t,\beta'} d\beta' \\ &= \frac{1}{2} \int \frac{1}{2} \left\{ -\partial_{\beta'} \left[(X_t(\alpha') - X_t(\beta'))^2 + (Y_t(\alpha') - Y_t(\beta'))^2 \right] \right\} \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) d\beta' \\ &= \frac{\pi}{8} \int \frac{(X_t(\alpha') - X_t(\beta'))^2 + (Y_t(\alpha') - Y_t(\beta'))^2}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} d\beta' \\ &\geq 0. \end{aligned} \quad (46)$$

Here, there is no boundary term in the integration by parts because $\bar{Z}_t|_\partial = 0$. Therefore

$$A_1 \geq 1. \quad (47)$$

3.2.2 Degenerate Taylor stability criterion and the singularity of the surface

We can draw a few important conclusions from the derivations in §3.2.1. For the sake of exposition, we will in this section focus on the angle ν at the wall, and we will move the corner from ± 1 to 0; angled crests and other singularities in the middle of the surface can be handled similarly. Let ν be the angle at the corner. Then the Riemann mapping $\Phi(z) \approx z^r$ at the corner, where $r\nu = \frac{\pi}{2}$. By (32), $Z_{,\alpha'} = (\Phi^{-1})_{z'}$. Therefore,

$$Z_{,\alpha'} = \partial_{\alpha'} \Phi^{-1} \approx (\alpha')^{1/r-1} \quad (48)$$

at the corner.

We first note that by (39) and (45)

$$\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}. \quad (49)$$

Therefore in the regime where the acceleration $|Z_{tt}| < \infty$, $Z_{,\alpha'} \neq 0$, by (47). This implies $r \geq 1$ and $\nu \leq \pi/2$. Similarly, this implies that angled crests have an angle $\leq \pi$.¹⁷

Now, because

$$A_1 \circ h = \frac{\mathfrak{a} |z_\alpha|^2}{h_\alpha} \quad (50)$$

and

$$\frac{-\partial P}{\partial \mathbf{n}} = |z_\alpha| \mathfrak{a} = \frac{A_1 \circ h}{|Z_{,\alpha'} \circ h|}, \quad (51)$$

$\frac{-\partial P}{\partial \mathbf{n}} \geq 0$ always holds. In the setup where the free surface is $C^{1,\gamma}$ and makes a right angle at the corner ($\nu = \pi/2$), $0 < c_0 \leq |(\Phi^{-1})_{z'}| \leq C_0 < \infty$. This together with the estimate $A_1 \geq 1$ gave [Wu97] a strictly positive lower bound for the Taylor coefficient $-\frac{\partial P}{\partial \mathbf{n}}$. In our situation, $\frac{1}{Z_{,\alpha'}} \rightarrow 0$ at the corner if $\nu < \frac{\pi}{2}$; similarly, $\frac{1}{Z_{,\alpha'}} = 0$ at an angled crest if the interior angle is $< \pi$. If A_1 is in addition bounded from above—which will be true when our energy is finite—we know that the degeneracy of $-\frac{\partial P}{\partial \mathbf{n}}$ corresponds precisely to the degeneracy of the Riemann mapping Φ . Because the degeneracy of the Riemann mapping corresponds precisely to the singularity of the free surface, we can conclude that the degeneracy of the Taylor coefficient $-\frac{\partial P}{\partial \mathbf{n}}$ corresponds precisely to the singularities of the free surface, whether they are angled crests in the middle or non-right angles at the walls.

We note also that $\mathfrak{a} \geq 0$; that, by (51), \mathfrak{a} degenerates alongside $-\frac{\partial P}{\partial \mathbf{n}}$; and that $\mathfrak{a} \approx h_\alpha \approx \frac{h_\alpha}{A_1 \circ h}$ by (50), provided A_1 and $|z_\alpha|$ are bounded away from 0 and ∞ .

Finally, we note that our spatial derivative $D_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'}$ is more regular, in a sense, than $\partial_{\alpha'}$. We know $\frac{1}{Z_{,\alpha'}} \rightarrow 0$ at singularities, which indicates that the weight function $\frac{1}{Z_{,\alpha'}}$ has some “regularizing” effect. Indeed, we have $D_{\alpha'} \bar{Z}_t, D_{\alpha'} \bar{Z}_{tt} \in L^\infty$, but $\partial_{\alpha'} \bar{Z}_t, \partial_{\alpha'} \bar{Z}_{tt}$ are only in L^2 in our energy space, for example; see §5.1.

3.2.3 \mathcal{A}_t

Now we seek a formula for the nonlinearity on the RHS of (35), \mathcal{A}_t . As in §3.2.1, we will hope that the LHS is close to holomorphic and apply $(I - \mathbb{H})$ to both sides and then invert. Here we start by multiplying both sides of (35) by $Z_{,\alpha'}$, since we want the RHS to be purely imaginary:

$$Z_{,\alpha'} (\bar{Z}_{ttt} + i \mathcal{A}_t \bar{Z}_{t,\alpha'}) = -i \mathcal{A}_t |Z_{,\alpha'}|^2. \quad (52)$$

We once again carefully expand the LHS. As before, let $F(z(\alpha, t), t) = \bar{z}_t(\alpha, t)$. Again, we have

$$\bar{z}_{tt} = (F_z \circ z) z_t + F_t \circ z, \quad (53)$$

¹⁷In fact, the angle ν at the corner is no more than 90° always, even if $|Z_{tt}| = \infty$. Indeed, if $|Z_{tt}| = \infty$ at the corner, then y_{tt} must be infinity at the corner, because $x_{tt} = 0$ at the corner. This implies, by (20) and (21), that $\nu = \frac{\pi}{2}$. Note that this argument applies only at the corner; for angled crests in the middle, we need $|Z_{tt}| < \infty$, which will hold in our energy regime.

so

$$\bar{z}_{ttt} = (F_{zz} \circ z)z_t^2 + 2(F_{tz} \circ z)z_t + (F_z \circ z)z_{tt} + F_{tt} \circ z. \quad (54)$$

We now solve for $F_z \circ z$, $F_{zz} \circ z$ and $F_{tz} \circ z$. Since $\partial_z = D_\alpha$ on holomorphic functions,

$$F_z \circ z = D_\alpha \bar{z}_t, \quad F_{zz} \circ z = D_\alpha^2 \bar{z}_t. \quad (55)$$

We solve for $F_{tz} \circ z$ by applying $\partial_z = D_\alpha$ to (53):

$$F_{tz} \circ z = D_\alpha (\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t). \quad (56)$$

Therefore, by substituting (55) and (56) into (54), we get

$$\bar{z}_{ttt} = (D_\alpha^2 \bar{z}_t)z_t^2 + 2z_t D_\alpha (\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t) + (D_\alpha \bar{z}_t)z_{tt} + F_{tt} \circ z. \quad (57)$$

Precomposing with h^{-1} , we have

$$\bar{Z}_{ttt} = (D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z. \quad (58)$$

We now go back to (52), substituting in (58) to get

$$Z_{,\alpha'} ((D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z + i\mathcal{A}_{\bar{Z}_{t,\alpha'}}) = -i\mathcal{A}_t |Z_{,\alpha'}|^2. \quad (59)$$

We simplify, distributing the $Z_{,\alpha'}$ and then using the identity $Z_{tt} + i = i\mathcal{A}Z_{,\alpha'}$ on the last term:

$$(\partial_{\alpha'} D_{\alpha'} \bar{Z}_t)Z_t^2 + 2Z_t \partial_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + 2\bar{Z}_{t,\alpha'} Z_{tt} + Z_{,\alpha'} (F_{tt} \circ Z) + i\bar{Z}_{t,\alpha'} = -i\mathcal{A}_t |Z_{,\alpha'}|^2. \quad (60)$$

We now apply $(I - \mathbb{H})$ to both sides. Various terms will disappear on the LHS and others will turn into commutators, due to holomorphicity; specifically, we use (370), (375), and (373). We get

$$\begin{aligned} [Z_t^2, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \bar{Z}_t + 2[Z_t, \mathbb{H}] \partial_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} \\ = (I - \mathbb{H}) \left\{ -i\mathcal{A}_t |Z_{,\alpha'}|^2 \right\}. \end{aligned} \quad (61)$$

We could continue working with this equation, but two integrations by parts will give us a nicer equation to work with. We take the first term and the second part of the second term and integrate by parts both terms, noting that we have no boundary terms. We get

$$[Z_t^2, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \bar{Z}_t - 2[Z_t, \mathbb{H}] \partial_{\alpha'} ((D_{\alpha'} \bar{Z}_t)Z_t) = -\frac{\pi}{4i} \int \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \bar{Z}_t(\beta') d\beta'. \quad (62)$$

This is a type of higher-order Calderon commutator, which we write as $-[Z_t, Z_t; D_{\alpha'} \bar{Z}_t]$ (see (11)). We therefore can rewrite (61) as

$$-i(I - \mathbb{H}) \left\{ \mathcal{A}_t |Z_{,\alpha'}|^2 \right\} = 2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t]. \quad (63)$$

Taking imaginary parts, we get

$$\mathcal{A}_t |Z_{,\alpha'}|^2 = -\Im (2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t]). \quad (64)$$

3.2.4 Formulas for $\frac{\mathfrak{a}_t}{\mathfrak{a}}$ and $\frac{\mathcal{A}_t}{\mathcal{A}}$

We could now plug (64) into (35) to get the quasilinear equation in Riemannian coordinates. Instead, though, we will focus on a different quantity. Observe that dividing (64) by (45) we have

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{\mathcal{A}_t}{\mathcal{A}} = \frac{\mathcal{A}_t |Z_{,\alpha'}|^2}{\mathcal{A} |Z_{,\alpha'}|^2} = \frac{-\Im (2[Z_t, \mathbb{H}] \bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t])}{A_1}. \quad (65)$$

If we want to control $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty}$, then, because $A_1 \geq 1$ by (47), it suffices to control the numerator of the RHS of (65), which we will be able to do.

4 The main result

4.1 Definition of the energy

We now introduce the energy for which we will prove an a priori inequality. We consider a general equation of the form

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\theta = G_\theta \quad (66)$$

with the constraint that θ is the boundary value of a periodic holomorphic function on $\Omega(t)$. Our base case is (19), written in this form as

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\bar{z}_t = -i\mathfrak{a}_t\bar{z}_\alpha, \quad (67)$$

with $\theta = \bar{z}_t$ and $G_{\bar{z}_t} = -i\mathfrak{a}_t\bar{z}_\alpha$. Higher-order cases will come from applying D_α^k to (19); in those cases, $\theta = D_\alpha^k\bar{z}_t$ with

$$G_\theta = D_\alpha^k(-i\mathfrak{a}_t\bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^k]\bar{z}_t. \quad (68)$$

A natural energy for (66) would be

$$\int |\theta_t|^2 d\alpha + \Re \int i(\mathfrak{a}\partial_\alpha\theta)\bar{\theta}d\alpha. \quad (69)$$

Here, the fact that θ is holomorphic allows the second term to be rewritten as $\int i(\mathfrak{a}\mathfrak{H}\partial_\alpha\theta)\bar{\theta}d\alpha$, where \mathfrak{H} is the Hilbert transform for the curved domain Ω . Because $i\mathfrak{H}\partial_\alpha$ is a positive operator—in flattened coordinates, it corresponds to $|D| = \sqrt{-\Delta}$ —the second term will be positive up to an error term (due to the coefficient \mathfrak{a}).

We will, instead, use two variants of this energy, differing primarily in that we multiply or divide by the roughly equivalent singular weights \mathfrak{a} and $\frac{h_\alpha}{A_1 \circ h}$. In earlier works, where there was a strict bound $\mathfrak{a} \geq c_0 > 0$ (in addition to the upper bound we continue to have), all of these energies were essentially equivalent, but here the singularity $\mathfrak{a} = 0$ makes an important difference, and so the choice of the weights is critical. We will discuss this further in §4.2 below.

We note that we are phrasing everything here in Lagrangian coordinates. With a change of variables, we can easily switch to Riemannian coordinates. We will often express our basic quantities in Lagrangian coordinates when we need to take a time derivative, but use Riemannian coordinates when we need to estimate terms, since that gives us access to the easily invertible $(I - \mathbb{H})$ operator.

Our first energy differs from (69) primarily by a multiplicative factor of h_α :

$$E_{a,\theta}(t) := \int_I |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I (i\mathfrak{a}\partial_\alpha\theta)\bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I \left(i \frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha + \int_I |\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha. \quad (70)$$

Here, the $A_1 \circ h$ appears for technical reasons, which will be explained below in §4.2. The first two terms are the main terms. The third term is a correction to the second term, to be explained below, and the fourth term is lower-order.

For our second energy, E_b , we divide through by \mathfrak{a} in the main terms:

$$E_{b,\theta}(t) := \int_I \frac{1}{\mathfrak{a}} |\theta_t|^2 d\alpha + \int_I (i\partial_\alpha\theta)\bar{\theta}d\alpha + \int_I \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 d\alpha. \quad (71)$$

Here, the first two integrals are the primary terms; the last is lower-order. It's easy to check, by integration by parts, that the second term is purely real.

Our total energy consists primarily of $E_{a,\theta}$ using two D_α derivatives and $E_{b,\theta}$ using one D_α derivative. In addition, we will include one other, lower-order term in our total energy: $|\bar{z}_{tt}(\alpha_0, t) - i|$ for some fixed $\alpha_0 \in I$. Our total energy therefore is

$$E = E(t) := E_{a,D_\alpha^2\bar{z}_t}(t) + E_{b,D_\alpha\bar{z}_t}(t) + |\bar{z}_{tt}(\alpha_0, t) - i|. \quad (72)$$

Even though we have now specialized to the case where $\theta = D_\alpha^2\bar{z}_t$ and $\theta = D_\alpha\bar{z}_t$ for $E_{a,\theta}$ and $E_{b,\theta}$, respectively, we will in the following present some of the proofs about these energies in broader generality. We will use $E_{a,\theta}$ and $E_{b,\theta}$ to refer to the generic first- and second-type energies, and use $E_a := E_{a,D_\alpha^2\bar{z}_t}$ and $E_b := E_{b,D_\alpha\bar{z}_t}$ for the specific energies.

4.2 Discussion

We offer a few comments on our energy.

We begin with E_a . On changing variables to Riemannian coordinates, the first and the fourth terms of E_a become $\|\theta_t \circ h^{-1}\|_{L^2(1/A_1)}^2$ and $\|\theta \circ h^{-1}\|_{L^2(1/A_1)}^2$. Since we will be able to control $\|A_1\|_{L^\infty}$ by our energy, these two terms therefore are equivalent to $\|\theta_t \circ h^{-1}\|_{L^2}$ and $\|\theta \circ h^{-1}\|_{L^2}$.

The second term and third terms in $E_{a,\theta}$ treated together equal $\left\| \frac{1}{\bar{Z}_{\alpha'}} (\theta \circ h^{-1}) \right\|_{\dot{H}^{1/2}}^2$. Indeed, when we use $A_1 \circ h = \frac{|z_\alpha|^2}{h_\alpha}$ (50) and then change variables, these two terms in $E_{a,\theta}$ become

$$\begin{aligned} \Re \int (i\alpha \partial_\alpha \theta) \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha + \Re \int_I \left(i \frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha &= \Re \int i \frac{h_\alpha^2}{|z_\alpha|^2} (\partial_\alpha \theta) \bar{\theta} d\alpha + \Re \int_I \left(i \frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\ &= \Re \int i \left(\partial_\alpha \left(\theta \frac{h_\alpha}{z_\alpha} \right) \right) \left(\bar{\theta} \frac{h_\alpha}{\bar{z}_\alpha} \right) d\alpha \\ &= \Re \int i \left(\partial_{\alpha'} \left(\frac{1}{\bar{Z}_{\alpha'}} (\theta \circ h^{-1}) \right) \right) \left(\frac{1}{\bar{Z}_{\alpha'}} (\bar{\theta} \circ h^{-1}) \right) d\alpha'. \end{aligned} \quad (73)$$

This equals the square of the $\dot{H}^{1/2}$ norm of $\frac{1}{\bar{Z}_{\alpha'}} (\theta \circ h^{-1})$; see §B.4. (Note that $\frac{1}{\bar{Z}_{\alpha'}} (\theta \circ h^{-1})$ is holomorphic.) This explains why we divided through by $A_1 \circ h$ in our definition of $E_{a,\theta}$. This $\dot{H}^{1/2}$ control will be crucial in allowing us to close our energy inequality.

We now specialize to $E_a = E_{a,D_\alpha^2 \bar{z}_t}$. By the above discussions,

$$E_a = \|(\partial_t D_\alpha^2 \bar{z}_t) \circ h^{-1}\|_{L^2(1/A_1)}^2 + \left\| \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}^2 + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2(1/A_1)}^2. \quad (74)$$

We will be able to control the commutator $[\partial_t, D_\alpha^2]$, so the first term corresponds roughly to $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2(1/A_1)}$ or $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$; we make this more precise in §5.1.

Similarly, by using the half-derivative and the fact that $\|D_\alpha \bar{z}_t\|_{L^2(\frac{1}{A_1 \circ h})} = \|\bar{Z}_{t,\alpha'}\|_{L^2}$ (by (50)),

$$E_b = \|\partial_t D_\alpha \bar{z}_t\|_{L^2(\frac{1}{A_1})}^2 + \|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{1/2}}^2 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2. \quad (75)$$

Again, on changing variables to Riemannian coordinates, we see that, modulo a commutator $[\partial_t, D_\alpha]$ and a factor of A_1 , $\|\partial_t D_\alpha \bar{z}_t\|_{L^2(\frac{1}{A_1})} \approx \|\bar{Z}_{tt,\alpha'}\|_{L^2}$; we make this more precise, again, in §5.1. We will see in (93) that $\|A_1\|_{L^\infty}$ is bounded by $\|\bar{Z}_{t,\alpha'}\|_{L^2}^2 + 1$. This explains our choice of the lower-order term in (75). We note that both $\|A_1\|_{L^\infty}$ and $\|Z_{t,\alpha'}\|_{L^2}$ are invariant with respect to the scaling $\alpha' \sim t^2$ of the water wave equations (1)-(5), (7) (with $\Upsilon = \emptyset$).

The combination of E_a and E_b , with their roughly inverse singular weights $\frac{h_\alpha}{A_1 \circ h}$ and $\frac{1}{A_1}$, will allow us to control the L^∞ norms of various useful quantities (via the weighted Sobolev inequality (385)).

Finally, we also include $|\bar{z}_{tt}(\alpha_0, t) - i|$ in our energy since it, along with control of $\|\bar{Z}_{tt,\alpha'}\|_{L^2}$, will allow us to control the lower-order term $\|\bar{Z}_{tt} - i\|_{L^\infty}$. We note that $\|\bar{Z}_{tt} - i\|_{L^\infty}$ is invariant with respect to the scaling $\alpha' \sim t^2$, and we remark that our estimate does not depend on control of $\|Z_t\|_{L^\infty}$.

In §10 we give a characterization of our energy in terms of only \bar{Z}_t and $\frac{1}{\bar{Z}_{\alpha'}}$ and their spatial derivatives, and we discuss what types of singularities are allowed by a finite energy.

4.3 The main result

We now state our main result. We assume that the solution is suitably regular so that all the calculations in our proof make sense.

Theorem 2. *[A priori estimate] There exists a polynomial $p = p(x)$ with universal coefficients such that for any solution of water wave equations (1)-(7) with $E(t) < \infty$ for all $t \in [0, T]$,*

$$\frac{d}{dt}E(t) \leq p(E(t)) \quad (76)$$

for all $t \in [0, T]$.

Here the water wave equations (1)-(7) are those defined on the symmetric domain $\Omega(t)$, with fixed boundary $\Upsilon = \{x = \pm 1\}$, as specified in §1.1. We do not state precise polynomial $p(x)$, but it can be calculated by carefully combining the estimates we make in the proof. Estimates for the individual parts of the energy E_a , E_b , and $|\bar{z}_{tt}(\alpha_0) - i|$ are listed in §4.4. Although we state the a priori inequality for the original equations (1)-(7) on $\Omega(t)$, our proof relies exclusively on the Lagrangian reduction to the equation (17) on the free surface $\Sigma(t)$, along with the accompanying statements about \bar{z}_t 's holomorphicity and periodicity.

4.4 The proof

We begin the proof by differentiating each component of $E(t)$ in time. For the two main energies, E_a and E_b , we then integrate by parts to arrive at a term $\partial_t^2 \theta + i\alpha \partial_\alpha \theta$ and use the basic equation $\partial_t^2 \theta + i\alpha \partial_\alpha \theta = G_\theta$ to replace it with G_θ . What remain to be estimated will be G_θ , along with several ancillary terms. We control those quantities in §5 through §9 in terms of a polynomial of the energy.

4.4.1 The estimate for E_a

We begin by differentiating E_a with respect to t .

We will work initially with general θ satisfying $\theta|_{\partial} = 0$, $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, and the basic equation (66), and then we will specialize to the $\theta = D_\alpha^2 \bar{z}_t$ ¹⁸ in our energies. The periodicity ensures that there is no boundary term in the integration by parts.

We use the fact that $\frac{ah_\alpha}{(A_1 \circ h)} = \frac{h_\alpha^2}{|z_\alpha|^2}$ (by (50)) in the following calculation.

$$\begin{aligned} \frac{d}{dt}E_{a,\theta}(t) &= \int (\theta_{tt}\bar{\theta}_t + \theta_t\bar{\theta}_{tt}) \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |\theta_t|^2 \frac{h_{t\alpha}}{A_1 \circ h} d\alpha - \int |\theta_t|^2 \frac{h_\alpha}{A_1 \circ h} \frac{(A_1 \circ h)_t}{A_1 \circ h} d\alpha \\ &\quad + \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha + \underbrace{\Re \int i \frac{h_\alpha^2}{|z_\alpha|^2} \theta_{t\alpha} \bar{\theta} d\alpha}_{-\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha + \Re \int \frac{h_\alpha^2}{|z_\alpha|^2} \theta_t i \bar{\theta}_\alpha d\alpha} + \Re \int i \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta}_t d\alpha \\ &\quad + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) (\theta_t \bar{\theta} + \bar{\theta} \bar{\theta}_t) h_\alpha d\alpha + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_{t\alpha} d\alpha \\ &\quad + \int (\theta_t \bar{\theta} + \bar{\theta}_t \theta) \frac{h_\alpha}{A_1 \circ h} + \int |\theta|^2 \frac{h_{t\alpha}}{A_1 \circ h} d\alpha - \int |\theta|^2 \frac{h_\alpha}{A_1 \circ h} \frac{(A_1 \circ h)_t}{A_1 \circ h} d\alpha \\ &= \int 2\Re((\theta_{tt} + i\alpha \theta_\alpha) \bar{\theta}_t) \frac{h_\alpha}{A_1 \circ h} d\alpha + \int (|\theta_t|^2 + |\theta|^2) \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{(A_1 \circ h)_t}{A_1 \circ h} \right) \frac{h_\alpha}{A_1 \circ h} d\alpha \\ &\quad + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) (\theta_t \bar{\theta} + \bar{\theta} \bar{\theta}_t) h_\alpha d\alpha \\ &\quad + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 \frac{h_{t\alpha}}{h_\alpha} h_\alpha d\alpha + 2\Re \int \theta_t \bar{\theta} \frac{h_\alpha}{A_1 \circ h} d\alpha \\ &\quad - \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_t \bar{\theta} d\alpha + \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_\alpha \bar{\theta} d\alpha. \end{aligned} \quad (77)$$

¹⁸See (37).

Now we show how we control each of these terms.

For the first, we replace $\theta_{tt} + i\mathbf{a}\theta_\alpha$ with the RHS G_θ by the main equation (66) and then use Cauchy-Schwarz:

$$\int 2\Re((\theta_{tt} + i\mathbf{a}\theta_\alpha)\bar{\theta}_t) \frac{h_\alpha}{A_1 \circ h} d\alpha \lesssim \left(\int |G_\theta|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2} \left(\int |\bar{\theta}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}. \quad (78)$$

The first factor, which involves the RHS of the basic equation, is the main term to control. For $\theta = D_\alpha^2 \bar{z}_t$,

$$G_\theta = D_\alpha^2(-i\mathbf{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathbf{a}\partial_\alpha, D_\alpha^2] \bar{z}_t. \quad (79)$$

We estimate these terms in §9.

In §6, we will control

$$\left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha \right| \lesssim (161). \quad (80)$$

Because $\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} = D_{\alpha'} \frac{1}{Z_{,\alpha'}} \circ h$, we estimate

$$\left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) (\theta_t \bar{\theta} + \bar{\theta} \theta_t) h_\alpha d\alpha \right| \lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}. \quad (81)$$

Similarly, we estimate

$$\left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 \frac{h_{t\alpha}}{h_\alpha} h_\alpha d\alpha \right| \lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}. \quad (82)$$

We observe that

$$\left\| \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \right\|_{L^\infty} = \left\| \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|^2} \right\|_{L^\infty} \leq 2 \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}, \quad (83)$$

so, using Cauchy-Schwarz, we have

$$\left| -\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_\alpha \theta_t \bar{\theta} d\alpha \right| \lesssim \|A_1\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} E_{a,\theta}. \quad (84)$$

In §7 we control

$$\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \lesssim (187). \quad (85)$$

We estimate the remaining two terms of (77) by Cauchy-Schwarz and Hölder.

We now combine these estimates and specialize to $\theta = D_\alpha^2 \bar{z}_t$. Each of the remaining factors we will control in §5; we list the location of the final estimate for each quantity of the following in the subscripts. We get

$$\begin{aligned} \left| \frac{d}{dt} E_a \right| &\lesssim \underbrace{\|G_{D_\alpha^2 \bar{z}_t}\|_{L^2(\frac{h_\alpha}{A_1 \circ h})}}_{\lesssim(291)} E_a^{1/2} + \underbrace{\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}}_{\lesssim(129)} E_a + \underbrace{\left\| \frac{(A_1 \circ h)_t}{A_1 \circ h} \right\|_{L^\infty}}_{\lesssim(131)} E_a \\ &+ \underbrace{\left(1 + \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} \right)}_{\lesssim 1 + (129)} \|A_1\|_{L^\infty} \underbrace{\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}}_{\lesssim(141)} E_a + E_a \\ &+ \underbrace{\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha}_{\lesssim(161)} + \underbrace{\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha}_{\lesssim(187)}. \end{aligned} \quad (86)$$

4.4.2 The estimate for E_b

Now we consider our second energy. Once again, we work first with general θ satisfying $\theta|_{\partial} = 0$, $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$, and the main equation (66). Then we specialize to $\theta = D_\alpha \bar{z}_t$.¹⁹ The periodicity ensures there is no boundary term when we integrate by parts.

$$\begin{aligned}
\frac{d}{dt} E_{b,\theta}(t) &= \int \frac{1}{\mathfrak{a}} (\theta_{tt} \bar{\theta}_t + \theta_t \bar{\theta}_{tt}) d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{1}{\mathfrak{a}} |\theta_t|^2 d\alpha + \underbrace{\int i \theta_{t\alpha} \bar{\theta} d\alpha}_{= \int \overline{i \theta_\alpha} \theta_t d\alpha} + \int i \theta_\alpha \bar{\theta}_t d\alpha \\
&+ \int \frac{(A_1 \circ h)}{\mathfrak{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \frac{(A_1 \circ h)_t}{\mathfrak{a}} |\theta|^2 d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 d\alpha \\
&= 2\Re \int \frac{G_\theta}{\mathfrak{a}} \bar{\theta}_t d\alpha - \int \frac{\mathfrak{a}_t}{\mathfrak{a}} \frac{1}{\mathfrak{a}} |\theta_t|^2 d\alpha \\
&+ \int \frac{A_1 \circ h}{\mathfrak{a}} (\theta_t \bar{\theta} + \theta \bar{\theta}_t) d\alpha + \int \left(\frac{(A_1 \circ h)_t}{(A_1 \circ h)} - \frac{\mathfrak{a}_t}{\mathfrak{a}} \right) \frac{(A_1 \circ h)}{\mathfrak{a}} |\theta|^2 d\alpha.
\end{aligned} \tag{87}$$

By Hölder and Cauchy-Schwarz, we conclude that

$$\left| \frac{d}{dt} E_{b,\theta}(t) \right| \lesssim \left\| \frac{G_\theta}{\sqrt{\mathfrak{a}}} \right\|_{L^2} E_{b,\theta}^{1/2} + \left(\|A_1\|_{L^\infty}^{1/2} + \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty} \right) E_{b,\theta}. \tag{88}$$

For $\theta = D_\alpha \bar{z}_t$, we control $\left\| \frac{G_\theta}{\sqrt{\mathfrak{a}}} \right\|_{L^2}$ in §8, at (208). We control $\|A_1\|_{L^\infty}$ at (93), $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty}$ at (114) and $\left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty}$ at (131).

4.4.3 The proof for $|z_{tt}(\alpha_0, t) - i|$

Finally, we show that we can control $\frac{d}{dt} |\bar{z}_{tt}(\alpha_0, t) - i|$. By differentiating with respect to t , we have, by the basic quasilinear equation (19),

$$\begin{aligned}
\frac{d}{dt} |\bar{z}_{tt}(\alpha_0) - i| &\leq |\bar{z}_{ttt}(\alpha_0)| = |i \mathfrak{a}_t(\alpha_0) \bar{z}_\alpha(\alpha_0) + i \mathfrak{a}(\alpha_0) \bar{z}_{t\alpha}(\alpha_0)| \\
&\lesssim \left(\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \|D_\alpha \bar{z}_t\|_{L^\infty} \right) |\bar{z}_{tt}(\alpha_0) - i|.
\end{aligned} \tag{89}$$

We control $\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty}$ below at (114) and $\|D_\alpha \bar{z}_t\|_{L^\infty}$ at (95).

4.5 Outline of the remainder of the proof

In sections §5 through §9, we complete the proof of the a priori inequality (76).

In §5, we control various quantities that are necessary for our proof. In §5.1, we carefully list the basic quantities controlled by our energy. In §5.2-§5.8, we estimate various other quantities in terms of quantities already controlled in previous subsections of §5. These quantities include many of the quantities to be controlled in §4.4 above, and are used in the remainder of the proof. In appendix §D, we list and give references to all the quantities controlled in §5, which we then use, sometimes without citation, in §6 through §9.

In §6 and §7 we estimate the terms from (80) and (85) in the estimate of $\frac{d}{dt} E_a$ above. Finally, in §8 and §9 we conclude the estimates for $\frac{d}{dt} E_b$ and $\frac{d}{dt} E_a$, respectively, by controlling the G_θ terms, completing the proof.

The basic approach for many of the estimates is to try and use the fact that certain quantities are purely real-valued and others are holomorphic to express problematic terms as commutators involving the Hilbert

¹⁹See (37).

transform, and then use the commutator estimates from §B.3 to avoid loss of derivatives. The estimates are very tight, and sometimes convoluted. Among the reasons for the complexity are: we have very little regularity to work with and our energy has direct control of only a selected collection of (weighted) quantities; we are working with a weighted derivative $D_{\alpha'}$ that has to be commuted with \mathbb{H} and whose complex-valued weight makes inverting $(I - \mathbb{H})$ on real-valued functions more difficult; we have no positive lower bound for the Taylor coefficient $-\frac{\partial P}{\partial \mathbf{n}} = \mathbf{a} |z_\alpha|$; because our estimates are tight, we have to take care in using different estimates for different terms, including treating certain terms as commutators while keeping others in $(I - \mathbb{H})$ form; we must take care that the quantities in our commutators have appropriate periodic boundary behavior; and we sometimes have to carefully decompose certain quantities into their holomorphic and antiholomorphic projections. We will give enough details to facilitate reading.

We use $C(E)$ to indicate a universal polynomial of E , which may differ from line to line.

Throughout the remaining derivations, we will repeatedly rely on the identity (50)

$$A_1 \circ h = \frac{\mathbf{a} |z_\alpha|^2}{h_\alpha}. \quad (90)$$

5 Quantities controlled by our energy

Here we collect together many quantities that are controlled by our energy. In appendix §D, we give a list of all the quantities controlled in this section that are quoted without citation in future sections or subsections.

5.1 Basic quantities controlled by the energy

In this section, we present a list of basic quantities controlled by our energy. Because conjugations and commutations of ∂_t with D_α add complexity, we take care to list some of those estimates as well. We list all of the basic terms controlled here at (112) below.

We start with (74) and (75). E_a and E_b directly control

$$\|D_\alpha^2 \bar{z}_t\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)}, \quad \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2(\frac{1}{A_1} d\alpha')}, \quad \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}} \leq E_a^{1/2} \quad (91)$$

$$\left(\int |D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \leq \|\bar{Z}_{t,\alpha'}\|_{L^2} \leq E_b^{1/2}, \quad (92)$$

where we used $A_1 \geq 1$ (47) in (92). By (45), using the commutator estimate (407), and then (92), we have

$$\|A_1\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 + 1 \lesssim E_b + 1. \quad (93)$$

Thanks to (93), we can now control $\|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} = \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}$ by a polynomial of our energy E .

Now we control $\|D_\alpha \bar{z}_t\|_{L^\infty} = \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}$. We work in Riemannian coordinates and use the weighted Sobolev inequality (386) with weight $\omega = \frac{1}{|Z_{,\alpha'}|^2}$ (and $\varepsilon = 1$). Note that $f(D_{\alpha'} \bar{Z}_t)^2 = 0$ by (384). This gives

$$\begin{aligned} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} &\lesssim \left(\int |D_{\alpha'} \bar{Z}_t|^2 |Z_{,\alpha'}|^2 d\alpha' \right)^{1/2} + \left(\int |\partial_{\alpha'} D_{\alpha'} \bar{Z}_t|^2 \frac{1}{|Z_{,\alpha'}|^2} d\alpha' \right)^{1/2} \\ &= \|\bar{Z}_{t,\alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \leq E_b^{1/2} + \|A_1\|_{L^\infty}^{1/2} E_a^{1/2} \\ &\lesssim C(E). \end{aligned} \quad (94)$$

We conclude that

$$\|D_\alpha \bar{z}_t\|_{L^\infty} = \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} = \|D_\alpha \bar{z}_t\|_{L^\infty} = \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \lesssim C(E). \quad (95)$$

Now we use the commutator identity (413) to move the ∂_t inside the first term in E_b :

$$\begin{aligned} \left(\int |D_\alpha \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} &\leq \left(\int |\partial_t D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} + \left(\int |[\partial_t, D_\alpha] \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ &\leq E_b^{1/2} + \|D_\alpha z_t\|_{L^\infty} \left(\int |D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ &\lesssim C(E) \end{aligned} \quad (96)$$

by (95) and (92). By changing variables and by (90), we conclude from (96) and (93) that

$$\begin{aligned} \|\bar{Z}_{tt, \alpha'}\|_{L^2} &\lesssim \|A_1\|_{L^\infty}^{1/2} \left(\int |D_\alpha \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}} \right)^{1/2} \\ &\lesssim C(E). \end{aligned} \quad (97)$$

Note that $|D_\alpha^2 f| \neq |D_\alpha^2 \bar{f}|$. Nevertheless, for generic f , we can control $D_\alpha^2 f$ by $D_\alpha^2 \bar{f}$ in $L^2(h_\alpha d\alpha)$ norm, at the expense of some lower-order terms. For notational convenience, we define here

$$|D_\alpha| := \frac{1}{|z_\alpha|} \partial_\alpha = \frac{z_\alpha}{|z_\alpha|} D_\alpha. \quad (98)$$

We expand:

$$\begin{aligned} D_\alpha^2 f &= \left(\frac{|z_\alpha|}{z_\alpha} \right)^2 |D_\alpha|^2 f + \frac{|z_\alpha|}{z_\alpha} \left(|D_\alpha| \frac{|z_\alpha|}{z_\alpha} \right) |D_\alpha| f \\ D_\alpha^2 \bar{f} &= \left(\frac{|z_\alpha|}{z_\alpha} \right)^2 |D_\alpha|^2 \bar{f} + \frac{|z_\alpha|}{z_\alpha} \left(|D_\alpha| \frac{|z_\alpha|}{z_\alpha} \right) |D_\alpha| \bar{f}. \end{aligned} \quad (99)$$

Therefore,

$$|D_\alpha^2 f| \leq |D_\alpha^2 \bar{f}| + 2 \left| |D_\alpha| \frac{|z_\alpha|}{z_\alpha} \right| |D_\alpha \bar{f}| \quad (100)$$

and so

$$\left(\int |D_\alpha^2 f|^2 h_\alpha d\alpha \right)^{1/2} \leq \left(\int |D_\alpha^2 \bar{f}|^2 h_\alpha d\alpha \right)^{1/2} + 2 \|D_\alpha \bar{f}\|_{L^\infty} \left(\int \left| D_\alpha \frac{|z_\alpha|}{z_\alpha} \right|^2 h_\alpha d\alpha \right)^{1/2}. \quad (101)$$

By (392) and then (391), (17), (90), and the fact $A_1 \geq 1$,

$$\left| D_\alpha \frac{|z_\alpha|}{z_\alpha} \right|^2 h_\alpha = \left| D_\alpha \frac{\bar{z}_{tt} - i}{|\bar{z}_{tt} - i|} \right|^2 h_\alpha \leq |D_\alpha \bar{z}_{tt}|^2 \frac{h_\alpha}{|\bar{z}_{tt} - i|^2} = |D_\alpha \bar{z}_{tt}|^2 \frac{h_\alpha}{\mathbf{a}^2 |z_\alpha|^2} \leq |D_\alpha \bar{z}_{tt}|^2 \frac{1}{\mathbf{a}}. \quad (102)$$

Plugging this into (101), and using (96), we get

$$\left(\int |D_\alpha^2 f|^2 h_\alpha d\alpha \right)^{1/2} \lesssim \left(\int |D_\alpha^2 \bar{f}|^2 h_\alpha d\alpha \right)^{1/2} + \|D_\alpha \bar{f}\|_{L^\infty} C(E). \quad (103)$$

We now apply (103) to $f = z_t$, using (95) to control $\|D_\alpha \bar{z}_t\|_{L^\infty}$ and (91) and (93) to control $\|D_\alpha^2 \bar{Z}_t\|_{L^2}$:

$$\|D_\alpha^2 Z_t\|_{L^2} = \left(\int |D_\alpha^2 z_t|^2 h_\alpha d\alpha \right)^{1/2} \lesssim C(E). \quad (104)$$

We now control $\|D_\alpha^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)}$. We use the commutator identity (414) to get

$$\begin{aligned}
\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} &= \|D_\alpha^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)} \\
&\leq \|\partial_t D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + 2 \|(D_\alpha z_t) D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + \|(D_\alpha^2 z_t) D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\
&\leq \|A_1\|_{L^\infty} E_a^{1/2} + 2 \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + \|D_\alpha^2 z_t\|_{L^2(h_\alpha d\alpha)} \|D_\alpha \bar{z}_t\|_{L^\infty} \\
&\lesssim C(E).
\end{aligned} \tag{105}$$

We will also need to control $D_{\alpha'}^2 Z_{tt}$; we delay doing this until later, after we control $\|D_\alpha \bar{z}_{tt}\|_{L^\infty}$.

We will also at one point need to control

$$\begin{aligned}
\|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} &\leq \|\partial_t D_\alpha D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + \|[\partial_t, D_\alpha] D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\
&\leq \|A_1\|_{L^\infty} E_a^{1/2} + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\
&\leq C(E).
\end{aligned} \tag{106}$$

We now control $\|\bar{z}_{tt} - i\|_{L^\infty} = \|\bar{Z}_{tt} - i\|_{L^\infty}$. Recall from our definition of the energy (72) that the energy includes $|\bar{z}_{tt}(\alpha_0, t) - i|$ for some fixed $\alpha_0 \in I$. Let $\alpha'_0 = h(\alpha_0, t)$. Then, by the fundamental theorem of calculus, for arbitrary $\alpha' \in I$,

$$\begin{aligned}
|\bar{z}_{tt}(\alpha', t) - i| &\lesssim |\bar{Z}_{tt}(\alpha'_0, t) - i| + \|\bar{Z}_{tt, \alpha'}\|_{L^1(I)} \\
&\lesssim |\bar{Z}_{tt}(\alpha'_0, t) - i| + (97).
\end{aligned} \tag{107}$$

We conclude that

$$\begin{aligned}
\|z_{tt} + i\|_{L^\infty} &= \|Z_{tt} + i\|_{L^\infty} = \|\bar{z}_{tt} - i\|_{L^\infty} = \|\bar{Z}_{tt} - i\|_{L^\infty} \\
&\lesssim C(E).
\end{aligned} \tag{108}$$

Because of (49) and (47), we can also conclude that

$$\left\| \frac{1}{\bar{Z}_{, \alpha'}} \right\|_{L^\infty} \lesssim C(E). \tag{109}$$

We use this to control $\|D_\alpha \bar{z}_{tt}\|_{L^\infty}$, using the weighted Sobolev inequality (385) in Riemannian coordinates with weight $\omega = \frac{1}{|\bar{Z}_{, \alpha'}|^2}$ (and $\varepsilon = 1$):²⁰

$$\begin{aligned}
\|D_\alpha z_{tt}\|_{L^\infty} &= \|D_{\alpha'} Z_{tt}\|_{L^\infty} = \|D_\alpha \bar{z}_{tt}\|_{L^\infty} = \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} \\
&\lesssim \|\bar{Z}_{tt, \alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} + \left(\int |D_{\alpha'} \bar{Z}_{tt}|^2 d\alpha' \right)^{1/2} \\
&\lesssim (1 + \|1/\bar{Z}_{, \alpha'}\|_{L^\infty}) \|\bar{Z}_{tt, \alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} \\
&\lesssim (1 + (109)) (97) + (105) \\
&\lesssim C(E).
\end{aligned} \tag{110}$$

Finally, we use (103), (105), and (110) to control $D_\alpha^2 z_{tt}$ and $D_{\alpha'}^2 Z_{tt}$:

$$\begin{aligned}
\|D_{\alpha'}^2 Z_{tt}\|_{L^2} &= \|D_\alpha^2 z_{tt}\|_{L^2(h_\alpha d\alpha)} \\
&\lesssim \|D_{\alpha'}^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)} + \|D_{\alpha'} \bar{z}_{tt}\|_{L^\infty} C(E) \\
&\lesssim (105) + (110) C(E) \lesssim C(E).
\end{aligned} \tag{111}$$

²⁰Note that unlike our proof for $\|D_{\alpha'} \bar{Z}_t\|_{L^\infty}$ at (94) above, we don't necessarily have that $f(D_{\alpha'} \bar{Z}_{tt})^2$ is zero, so we get a third term in the Sobolev inequality.

To sum up, we have the following quantities and their counterparts in Lagrangian coordinates controlled by universal polynomials of the energy E :

$$\begin{aligned}
& \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \|D_{\alpha'}^2 Z_{tt}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \|D_{\alpha'}^2 Z_t\|_{L^2}, \|D_{\alpha'} \partial_t D_{\alpha'} \bar{z}_t\|_{L^2(h_{\alpha} d\alpha)}, \\
& \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}, \|D_{\alpha'} Z_{tt}\|_{L^\infty}, \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}, \|D_{\alpha'} Z_t\|_{L^\infty}, \\
& \|\bar{Z}_{tt,\alpha'}\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \int |D_{\alpha'} \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}}, \int |D_{\alpha'} \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}}, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}, \|Z_{tt} + i\|_{L^\infty}, \|A_1\|_{L^\infty} \\
& \lesssim C(E).
\end{aligned} \tag{112}$$

5.2 Controlling $\left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty}$

We now show that we can control $\left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty}$, using (65). Because $A_1 \geq 1$ (47), it suffices to control

$$\|2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\bar{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'} \bar{Z}_t]\|_{L^\infty}. \tag{113}$$

We control the first two terms by (407), and the last term by Hölder and then Hardy's inequality (393). We have

$$\left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \|\bar{Z}_{tt,\alpha'}\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2}^2 \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \lesssim C(E). \tag{114}$$

5.3 Controlling $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$

Recall from (49) that $\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}$. Therefore,

$$\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt,\alpha'}}{A_1} - i \frac{\bar{Z}_{tt} - i}{A_1^2} \partial_{\alpha'} A_1. \tag{115}$$

Because $A_1 \geq 1$ (47), we can control the first term by $\|\bar{Z}_{tt,\alpha'}\|_{L^2}$. Now we address the second term.

We recall that $A_1 = \Im(-[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'}) + 1$ (45). Therefore,

$$\begin{aligned}
\partial_{\alpha'} A_1 &= \partial_{\alpha'} \Im \frac{-1}{2i} \int (Z_t(\alpha') - Z_t(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \bar{Z}_{t,\beta'} d\beta' \\
&= - \underbrace{\Im Z_{t,\alpha'} \mathbb{H} \bar{Z}_{t,\alpha'}}_0 + \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \bar{Z}_{t,\beta'}(\beta') d\beta \\
&= \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \bar{Z}_{t,\beta'}(\beta') d\beta,
\end{aligned} \tag{116}$$

where the first term disappears because $\mathbb{H} \bar{Z}_{t,\alpha'} = \bar{Z}_{t,\alpha'}$ (370) and so $Z_{t,\alpha'} \mathbb{H} \bar{Z}_{t,\alpha'}$ is purely real. Therefore, multiplying (116) by $|\bar{Z}_{tt}(\alpha) - i|$ and splitting into two parts, we have

$$\begin{aligned}
|\bar{Z}_{tt} - i| \partial_{\alpha'} A_1 &= \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (|\bar{Z}_{tt}(\alpha') - i| - |\bar{Z}_{tt}(\beta') - i|) \bar{Z}_{t,\beta'}(\beta') d\beta \\
&\quad + \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} |\bar{Z}_{tt}(\beta') - i| \bar{Z}_{t,\beta'}(\beta') d\beta \\
&= I + II.
\end{aligned} \tag{117}$$

We need to control $\|I\|_{L^2}$ and $\|II\|_{L^2}$. By (404),

$$\|I\|_{L^2} \lesssim \|Z_{t,\alpha}\|_{L^2} \|\bar{Z}_{tt,\alpha}\|_{L^2} \|\bar{Z}_{t,\alpha'}\|_{L^2} = \|\bar{Z}_{t,\alpha'}\|_{L^2}^2 \|\bar{Z}_{tt,\alpha'}\|_{L^2}. \tag{118}$$

For II , we replace $|\overline{Z}_{tt}(\beta') - i|$ by $\left| \frac{(-iA_1(\beta'))}{\overline{Z}_{,\beta'}} \right|$ (49) and use estimate (397), noticing that $\frac{1}{\overline{Z}_{,\beta'}} \overline{Z}_{t,\beta'} = D_{\alpha'} \overline{Z}_t$:

$$\|II\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2} \|A_1 D_{\alpha'} \overline{Z}_t\|_{L^\infty} \leq \|Z_{t,\alpha'}\|_{L^2} \|D_{\alpha'} \overline{Z}_t\|_{L^\infty} \|A_1\|_{L^\infty}. \quad (119)$$

We conclude that

$$\begin{aligned} \|(\overline{Z}_{tt} - i) \partial_{\alpha'} A_1\|_{L^2} &\lesssim \|\overline{Z}_{tt,\alpha'}\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}^2 + \|Z_{t,\alpha'}\|_{L^2} \|D_{\alpha'} \overline{Z}_t\|_{L^\infty} \|A_1\|_{L^\infty} \\ &\lesssim C(E) \end{aligned} \quad (120)$$

and

$$\begin{aligned} \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2} &\lesssim \|\overline{Z}_{tt,\alpha'}\|_{L^2} + \|(\overline{Z}_{tt} - i) \partial_{\alpha'} A_1\|_{L^2} \\ &\lesssim C(E). \end{aligned} \quad (121)$$

5.4 Controlling $\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}$

We recall (28): $h(\alpha, t) = \Phi(z(\alpha, t), t) = \Phi \circ z$. Therefore $h_\alpha = (\Phi_z \circ z) z_\alpha$, and

$$h_t = (\Phi_t \circ z) + (\Phi_z \circ z) z_t = (\Phi_t \circ z) + \frac{h_\alpha}{z_\alpha} z_t. \quad (122)$$

We precompose with h^{-1} :

$$(h_t \circ h^{-1})(\alpha', t) = \Phi_t \circ Z + \frac{1}{\overline{Z}_{,\alpha'}} Z_t. \quad (123)$$

Differentiating with respect to α' gives

$$\begin{aligned} (h_t \circ h^{-1})_{\alpha'} &= \partial_{\alpha'}(\Phi_t \circ Z) + D_{\alpha'} Z_t + Z_t \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \\ &= \partial_{\alpha'}(\Phi_t \circ Z) + 2\Re D_{\alpha'} Z_t - \frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}} + Z_t \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}}. \end{aligned} \quad (124)$$

We replaced $D_{\alpha'} Z_t$ by $2\Re D_{\alpha'} Z_t - \frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}}$, since we can turn $(I - \mathbb{H})(\frac{\overline{Z}_{t,\alpha'}}{\overline{Z}_{,\alpha'}})$ into a controllable commutator. Observe that the LHS and the second term on the RHS are purely real. Therefore, if we apply $\Re(I - \mathbb{H})$ to both sides, the LHS and the second term on the RHS remain unchanged, the term $\Re(I - \mathbb{H}) \partial_{\alpha'}(\Phi_t \circ Z)$ on the RHS disappears by (377), and the remaining terms become commutators by (370) and (368):

$$\begin{aligned} (h_t \circ h^{-1})_{\alpha'} &= 2\Re D_{\alpha'} Z_t + \Re \left\{ -(I - \mathbb{H}) \left(\frac{1}{\overline{Z}_{,\alpha'}} \overline{Z}_{t,\alpha'} \right) + (I - \mathbb{H}) Z_t \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\} \\ &= 2\Re D_{\alpha'} Z_t + \Re \left\{ - \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\}. \end{aligned} \quad (125)$$

We use (407) to control these:

$$\left\| \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H} \right] \overline{Z}_{t,\alpha'} \right\|_{L^\infty} \lesssim \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2} \|\overline{Z}_{t,\alpha'}\|_{L^2} \quad (126)$$

$$\left\| [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{L^2}. \quad (127)$$

Observe that

$$\frac{h_{t\alpha}}{h_\alpha} \circ h^{-1} = \partial_{\alpha'}(h_t \circ h^{-1}). \quad (128)$$

We therefore conclude that

$$\begin{aligned} \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} &= \|\partial_{\alpha'}(h_t \circ h^{-1})\|_{L^\infty} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|\bar{Z}_{t,\alpha'}\|_{L^2} \\ &\lesssim C(E). \end{aligned} \quad (129)$$

5.5 Controlling $\left\| \frac{(A_1 \circ h)_t}{(A_1 \circ h)} \right\|_{L^\infty}$

Recall that $A_1 \circ h = \frac{\mathfrak{a}|z_\alpha|^2}{h_\alpha}$ (50). Therefore,

$$\frac{\frac{d}{dt}(A_1 \circ h)}{A_1 \circ h} = \frac{\mathfrak{a}_t}{\mathfrak{a}} - \frac{h_{t\alpha}}{h_\alpha} + 2\Re D_\alpha z_t. \quad (130)$$

We have controlled each of the terms on the RHS in L^∞ in the previous sections. We conclude that

$$\left\| \frac{\frac{d}{dt}(A_1 \circ h)}{A_1 \circ h} \right\|_{L^\infty} \lesssim \left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \right\|_{L^\infty} + \left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \lesssim C(E). \quad (131)$$

5.6 Controlling $\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ and $\left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$

Recall from (49) that $\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}$. Therefore,

$$D_{\alpha'} \frac{1}{Z_{,\alpha'}} = i \frac{D_{\alpha'} \bar{Z}_{tt}}{A_1} - i \frac{\bar{Z}_{tt} - i}{A_1^2} D_{\alpha'} A_1 = i \frac{D_{\alpha'} \bar{Z}_{tt}}{A_1} + \frac{(\bar{Z}_{tt} - i)^2}{A_1^3} \partial_{\alpha'} A_1. \quad (132)$$

Because we can control the first term on the RHS by $\|D_{\alpha'} Z_{tt}\|_{L^\infty}$, it suffices to focus on the second term. We start from (116), and then use a similar idea to (117):

$$\begin{aligned} |\bar{Z}_{tt} - i|^2 \partial_{\alpha'} A_1 &= \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (|\bar{Z}_{tt}(\alpha') - i|^2 - |\bar{Z}_{tt}(\beta') - i|^2) \bar{Z}_{t,\beta'}(\beta') d\beta \\ &\quad + \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} |\bar{Z}_{tt}(\beta') - i|^2 \bar{Z}_{t,\beta'}(\beta') d\beta \\ &= I + II. \end{aligned} \quad (133)$$

To control $\|I\|_{L^\infty}$, we use the mean value theorem and the periodicity of Z_{tt} (346) to estimate

$$\left| \frac{|\bar{Z}_{tt}(\alpha') - i|^2 - |\bar{Z}_{tt}(\beta') - i|^2}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \right| \lesssim \|(\bar{Z}_{tt}(\alpha') - i) \partial_{\alpha'} \bar{Z}_{tt}(\alpha')\|_{L^\infty} \lesssim \|A_1\|_{L^\infty} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}. \quad (134)$$

From Cauchy-Schwarz and Hardy's inequality (393), we get

$$\|I\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \|A_1\|_{L^\infty} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}. \quad (135)$$

For II , observe that $|\bar{Z}_{tt}(\alpha') - i|^2 \bar{Z}_{t,\alpha'}(\alpha') = \frac{A_1^2}{Z_{,\alpha'}} D_{\alpha'} \bar{Z}_t$ and $\frac{A_1^2}{Z_{,\alpha'}} D_{\alpha'} \bar{Z}_t \Big|_\partial = 0$.²¹ We integrate by parts as in (395):

$$\begin{aligned} II &= \Im \frac{1}{2i} \int \frac{\pi}{2} \frac{(Z_t(\alpha') - Z_t(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} |\bar{Z}_{tt}(\beta') - i|^2 \bar{Z}_{t,\beta'}(\beta') d\beta \\ &= \Im \mathbb{H} \left(Z_{t,\alpha'} \frac{A_1^2}{Z_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) - \Im [Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{A_1^2}{Z_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right). \end{aligned} \quad (136)$$

²¹By (347), (350), and the evenness of A_1 .

We estimate the second term by (407):

$$\left\| [Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2}. \quad (137)$$

We expand the first term, using the conjugate of (370), $\mathbb{H}Z_{t,\alpha'} = -Z_{t,\alpha'}$, and noticing that $\Im \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) = 0$:

$$\Im \mathbb{H} \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) = -\Im \left(Z_{t,\alpha'} \frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t + \left[\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t, \mathbb{H} \right] Z_{t,\alpha'} \right) = -\Im \left[\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t, \mathbb{H} \right] Z_{t,\alpha'}. \quad (138)$$

We estimate the RHS by (407):

$$\left\| \left[\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t, \mathbb{H} \right] Z_{t,\alpha'} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2}. \quad (139)$$

Now,

$$\begin{aligned} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} &\lesssim \|A_1\|_{L^\infty}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \\ &\quad + \|A_1\|_{L^\infty}^2 \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} + \left\| \frac{A_1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} A_1 \right\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}. \end{aligned} \quad (140)$$

Combining the above calculations and using (120), (121), the estimates in §5.1, and the fact $A_1 \geq 1$, we conclude

$$\begin{aligned} \left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} &\leq \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|(\bar{Z}_{tt} - i)^2 \partial_{\alpha'} A_1\|_{L^\infty} \\ &\lesssim \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2}^2 \|A_1\|_{L^\infty} \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{A_1^2}{\bar{Z}_{,\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} \\ &\lesssim C(E). \end{aligned} \quad (141)$$

We record here the estimate for a related quantity, which we will use in later sections:

$$\left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \leq \|A_1\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \lesssim C(E). \quad (142)$$

5.7 Controlling $\|(I + \mathbb{H})D_{\alpha'} Z_t\|_{L^\infty}$

We note that this is the conjugate of the second term on the RHS of (125). We give details nevertheless. By taking a conjugate, using (370) to get a commutator, and using commutator estimate (407), we have

$$\begin{aligned} \|(I + \mathbb{H})D_{\alpha'} Z_t\|_{L^\infty} &= \left\| (I - \mathbb{H}) \frac{1}{\bar{Z}_{,\alpha'}} \bar{Z}_{t,\alpha'} \right\|_{L^\infty} = \left\| \left[\frac{1}{\bar{Z}_{,\alpha'}}, \mathbb{H} \right] \bar{Z}_{t,\alpha'} \right\|_{L^\infty} \\ &\lesssim \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \|\bar{Z}_{t,\alpha'}\|_{L^2} \lesssim C(E). \end{aligned} \quad (143)$$

5.8 Controlling $\left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty}$ and $\left\| (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\} \right\|_{L^\infty}$

In this section, we estimate $\left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty}$ and $\left\| (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\} \right\|_{L^\infty}$, which we will use in §9.

We know $\frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial} = 0$ by (346) and (350). By (339) we may commute in the derivative, so

$$\begin{aligned} \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} &= (I - \mathbb{H}) \partial_{\alpha'} \frac{Z_t}{Z_{,\alpha'}} \\ &= (I - \mathbb{H}) D_{\alpha'} Z_t + (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\}. \end{aligned} \quad (144)$$

First, we control the first term. Rewriting $(I - \mathbb{H}) = 2I - (I + \mathbb{H})$, we have

$$\|(I - \mathbb{H}) D_{\alpha'} Z_t\|_{L^\infty} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} + \|(I + \mathbb{H}) D_{\alpha'} Z_t\|_{L^\infty}, \quad (145)$$

where the second term was estimated at (143) above and the first term in §5.1.

We write the second term of (144) as a commutator by (368), and control this by (407):

$$\begin{aligned} \left\| (I - \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\} \right\|_{L^\infty} &= \left\| [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \\ &\lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim C(E). \end{aligned} \quad (146)$$

We conclude from (145), (146), and (143) that

$$\left\| \partial_{\alpha'}(I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim C(E). \quad (147)$$

6 Controlling $\Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha$

In this section, we control from (86) the term

$$\begin{aligned} \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha &= -\Re \int i \frac{\bar{z}_{t\alpha}}{\bar{z}_\alpha} \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \\ &\quad + \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \partial_t \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \end{aligned} \quad (148)$$

for $\theta = D_\alpha^2 \bar{z}_t$. We can control the first of these terms by

$$\left| -\Re \int i \frac{\bar{z}_{t\alpha}}{\bar{z}_\alpha} \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \right| \lesssim \|D_\alpha \bar{z}_t\|_{L^\infty} \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2. \quad (149)$$

Therefore, it suffices to focus on the second term on the RHS of (148). We expand it out:

$$\begin{aligned} \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \partial_t \frac{h_\alpha}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha &= \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \left(\frac{h_\alpha}{z_\alpha} \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) \right) |\theta|^2 h_\alpha d\alpha \\ &= \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha + \Re \int i \left(\frac{h_\alpha}{|z_\alpha|^2} \partial_\alpha \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha. \end{aligned} \quad (150)$$

We can estimate the first term on the RHS by

$$\left| \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) |\theta|^2 h_\alpha d\alpha \right| \lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}. \quad (151)$$

Therefore, it suffices to focus on the second term on the RHS of (150). Observe that because h is real-valued,

$$\Re \int i \left(\frac{h_\alpha}{|z_\alpha|^2} \partial_\alpha \left(\frac{h_{t\alpha}}{h_\alpha} - \frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha = -\Re \int i \left(\frac{h_\alpha}{|z_\alpha|^2} \partial_\alpha \left(\frac{z_{t\alpha}}{z_\alpha} \right) \right) |\theta|^2 h_\alpha d\alpha. \quad (152)$$

We now drop \Re and the i , write $D_{\alpha'}^2 \bar{z}_t = \theta$, and switch to Riemannian coordinates. For consistency with the quantities we've controlled elsewhere, we will take a conjugate. We have

$$\int \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha'. \quad (153)$$

We want to take advantage of the holomorphicity and antiholomorphicity of various of these factors. To do this, we first replace the $\frac{1}{\bar{Z}_{,\alpha'}}$ with a $\frac{1}{Z_{,\alpha'}}$ inside the first factor to make it closer to holomorphic:

$$\begin{aligned} & \int \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \\ &= \int \left(\frac{1}{|Z_{,\alpha'}|^2} \frac{Z_{,\alpha'}}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' + \int \left(\frac{1}{|Z_{,\alpha'}|^2} \left(\partial_{\alpha'} \frac{Z_{,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \left(\frac{\bar{Z}_{t,\alpha'}}{\bar{Z}_{,\alpha'}} \right) \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \quad (154) \\ &= \int \left(\frac{1}{\bar{Z}_{,\alpha'}^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' - \int \left(\frac{1}{|Z_{,\alpha'}|^2} \left(\partial_{\alpha'} \frac{Z_{tt} + i}{\bar{Z}_{tt} - i} \right) D_{\alpha'} \bar{Z}_t \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha', \end{aligned}$$

where we have used (34) to replace the $\frac{Z_{,\alpha'}}{\bar{Z}_{,\alpha'}}$ with $-\frac{Z_{tt}+i}{\bar{Z}_{tt}-i}$. We can estimate the second term by

$$\left| \int \left(\frac{1}{|Z_{,\alpha'}|^2} \left(\partial_{\alpha'} \frac{Z_{tt} + i}{\bar{Z}_{tt} - i} \right) D_{\alpha'} \bar{Z}_t \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' \right| \lesssim \|D_{\alpha'} Z_{tt}\|_{L^\infty} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2, \quad (155)$$

where we have used $\frac{1}{A_1} = \left| \frac{1}{Z_{,\alpha'}(Z_{tt}+i)} \right|$ (49), $A_1 \geq 1$ (47), and (391).

It therefore remains only to control the first term on the RHS of (154). Now we take advantage of holomorphicity. We rewrite this as

$$\int \left(\frac{1}{\bar{Z}_{,\alpha'}^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) |D_{\alpha'}^2 \bar{Z}_t|^2 d\alpha' = \int \left((\mathbb{P}_A + \mathbb{P}_H) \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \left(\frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \mathbb{P}_H D_{\alpha'}^2 \bar{Z}_t d\alpha', \quad (156)$$

where we have used (369) to insert \mathbb{P}_H in front of the $D_{\alpha'}^2 \bar{Z}_t$ and decomposed the first factor into the holomorphic and antiholomorphic projections. Now we use the adjoint property (337) to turn the \mathbb{P}_H into a \mathbb{P}_A on the opposite factors, and control using Cauchy-Schwarz:

$$\begin{aligned} & \left| \int \left((\mathbb{P}_A + \mathbb{P}_H) \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \left(\frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \mathbb{P}_H D_{\alpha'}^2 \bar{Z}_t d\alpha' \right| \\ & \lesssim \left\| \mathbb{P}_A \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \left(\frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\} \right\|_{L^2} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}. \end{aligned} \quad (157)$$

It now remains only to control this first factor.

First we consider the term with the \mathbb{P}_H . In this case, we can rewrite this as a commutator:

$$\begin{aligned} \mathbb{P}_A \left\{ \left(\mathbb{P}_H \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \left(\frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\} &= \frac{1}{2} \left[\frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t, \mathbb{H} \right] \mathbb{P}_H \left(\frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \\ &+ \frac{1}{4} \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \left(\int \frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right). \end{aligned} \quad (158)$$

Here, the mean term appears because of (336). We now use commutator estimate (403) for the first term

and Hölder for the second term, to conclude that

$$\begin{aligned}
& \left\| \mathbb{P}_A \left\{ \left(\mathbb{P}_H \left(\frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right) \left(\overline{\frac{1}{Z_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t} \right) \right\} \right\|_{L^2} \\
& \lesssim \left\| \overline{\frac{1}{Z_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t} \right\|_{\dot{H}^{1/2}} \left\| \mathbb{P}_H \left(\frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} + \left\| \frac{1}{Z_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \left\| \frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right\|_{L^1} \\
& \lesssim \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \left(\left\| \frac{1}{Z_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}} + \left\| \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \right).
\end{aligned} \tag{159}$$

Finally, we consider the \mathbb{P}_A term in the first factor on the RHS of (157). By the L^2 boundedness of \mathbb{P}_A , it suffices to control

$$\begin{aligned}
& \left\| \frac{1}{Z_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t (I - \mathbb{H}) \left(\frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} \\
& \lesssim \left\| D_{\alpha'}^2 \bar{Z}_t \left[\frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right\|_{L^2} + \left\| D_{\alpha'}^2 \bar{Z}_t (I - \mathbb{H}) \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \\
& \lesssim \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \left\| \left[\frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \right\|_{L^\infty} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \left\| \left[\frac{1}{\bar{Z}_{\alpha'}}, \mathbb{H} \right] D_{\alpha'}^2 \bar{Z}_t \right\|_{L^\infty} \\
& \lesssim \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2},
\end{aligned} \tag{160}$$

where we've used (369) to get the second commutator and used commutator estimate (407).

We now combine our estimates, concluding that

$$\begin{aligned}
& \Re \int i \left(\frac{1}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right)_t |\theta|^2 h_\alpha d\alpha \lesssim (149) + (150) \\
& \lesssim (149) + (151) + (155) + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \cdot ((159) + (160)) \\
& \lesssim C(E).
\end{aligned} \tag{161}$$

7 Controlling $\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha$

We now show that we can control the following term from the RHS of (86):

$$\Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha = \Re \int i \left(2 \frac{h_{t\alpha}}{h_\alpha} - 2 \Re D_\alpha z_t \right) \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta} d\alpha. \tag{162}$$

Here, all results will be expressed in terms of general energy $E_{\alpha, \theta}$ for θ satisfying $(I - \mathbb{H})(\theta \circ h^{-1}) = 0$ and $\theta|_\partial = 0$, rather than specifying $\theta = D_\alpha^2 \bar{z}_t$.²²

We begin by rewriting this as

$$\begin{aligned}
& \Re \int i \left(2 \frac{h_{t\alpha}}{h_\alpha} - 2 \Re D_\alpha z_t \right) \frac{h_\alpha^2}{|z_\alpha|^2} \theta_\alpha \bar{\theta} d\alpha = \Re \int 2i \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \left(\partial_\alpha \left(\theta \frac{h_\alpha}{z_\alpha} \right) \right) \bar{\theta} \frac{h_\alpha}{\bar{z}_\alpha} d\alpha \\
& \quad - \Re \int 2i \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \left(\frac{h_\alpha}{\bar{z}_\alpha} \partial_\alpha \frac{h_\alpha}{z_\alpha} \right) \theta \bar{\theta} d\alpha \\
& \quad = I + II.
\end{aligned} \tag{163}$$

²² $D_\alpha^2 \bar{z}_t$ satisfies $(I - \mathbb{H}) D_\alpha^2 \bar{z}_t \circ h^{-1} = 0$, $D_\alpha^2 \bar{z}_t|_\partial = 0$; see (37).

II is easy to control, via Hölder and change of variables to Riemannian coordinates:

$$|II| \lesssim \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \|A_1\|_{L^\infty} E_{a,\theta}. \quad (164)$$

Therefore, we can focus on I from (163). We introduce the following notations:

$$\psi := \left(\frac{h_\alpha}{z_\alpha} \theta \right) \circ h^{-1}; \quad \Theta := \theta \circ h^{-1}; \quad B := \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_\alpha z_t \right) \circ h^{-1} = (h_t \circ h^{-1})_{\alpha'} - \Re D_{\alpha'} Z_t. \quad (165)$$

We know $(I - \mathbb{H})\Theta = 0$, $\Theta|_\partial = 0$, and

$$B|_\partial = 0, \quad \psi|_\partial = 0 \quad (166)$$

$$(I - \mathbb{H})\psi = 0 \quad (167)$$

$$\|\psi\|_{\dot{H}^{1/2}} + \|A_1\|_{L^\infty}^{-1/2} \|\Theta\|_{L^2} \lesssim E_{a,\theta}^{1/2}, \quad (168)$$

where (166) follows from (348), (353), $\theta|_\partial = 0$, and (350); (167) follows from (362), $(I - \mathbb{H})\Theta = 0$ and principle no.2 in §A.3 (and for $\theta = D_\alpha^2 \bar{z}_t$ specifically from (371)); and (168) is immediate from the definition of $E_{a,\theta}$ and change of variables. Upon changing variables, we can write $I = \Re \int 2iB(\partial_{\alpha'}\psi)\bar{\psi}d\alpha'$.

Step 1. Green's identity. We now show that we can control I from (163). The main idea is to use Green's identity to move the derivative from $\psi := \left(\frac{h_\alpha}{z_\alpha} \theta \right) \circ h^{-1}$ onto B . We note that $i\partial_{\alpha'}\psi = i\partial_{\alpha'}\mathbb{H}\psi$ by (167), and that the operator $i\partial_{\alpha'}\mathbb{H}$ corresponds to the Dirichlet-Neumann operator ∇_n .²³ Letting ψ^h and B^h be the (periodic) harmonic extension of ψ and B to P^- respectively, we have

$$I = \Re \int 2iB(\partial_{\alpha'}\psi)\bar{\psi}d\alpha' = \Re \int 2B(\nabla_n\psi)\bar{\psi}d\alpha' = \int B\nabla_n(|\psi^h|^2)d\alpha'. \quad (169)$$

By Green's identity,²⁴

$$\begin{aligned} \int B\nabla_n(|\psi^h|^2)d\alpha' &= \int (\nabla_n B)|\psi|^2 d\alpha' + \int_{P^-} B^h \Delta(|\psi^h|^2)dv \\ &= I_1 + I_2. \end{aligned} \quad (170)$$

We control the second term, I_2 , by

$$\begin{aligned} |I_2| &= \left| \int_{P^-} B^h \Delta(|\psi^h|^2)dv \right| = 2 \left| \int_{P^-} B^h |\nabla \psi^h|^2 dv \right| \\ &\leq 2 \|B^h\|_{L^\infty} \int_{P^-} |\nabla \psi^h|^2 dv = \|B^h\|_{L^\infty} \int_{P^-} \Delta(|\psi^h|^2)dv \\ &= \|B\|_{L^\infty} \int \nabla_n(|\psi^h|^2)d\alpha' = 2 \|B\|_{L^\infty} \Re \int i(\partial_{\alpha'}\psi)\bar{\psi}d\alpha', \\ &= 2 \|B\|_{L^\infty} \|\psi\|_{\dot{H}^{1/2}} \lesssim \left(\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty} \right) \|\psi\|_{\dot{H}^{1/2}} \end{aligned} \quad (171)$$

by the maximum principle and another application of Green's identity.

Step 2. Controlling I_1 . We are left from Step 1 with controlling

$$\begin{aligned} I_1 &= \int (\nabla_n B)|\psi|^2 d\alpha' = \Re \int (i\partial_{\alpha'}\mathbb{H}B)|\psi|^2 d\alpha' \\ &= \Re \int (i\mathbb{H}\partial_{\alpha'}B)|\psi|^2 d\alpha' = \Re \int \frac{1}{Z_{,\alpha'}} (i\mathbb{H}\partial_{\alpha'}B)\Theta\bar{\psi}d\alpha', \end{aligned} \quad (172)$$

²³Recall that the Dirichlet-Neumann operator is defined by $\nabla_n f := \nabla_n f^h$, the outward-facing normal derivative of f^h , where f^h is the extension of f that is harmonic and periodic in P^- and tending to a constant at infinity. For f real-valued, we can derive this by noting that $(I + \mathbb{H})f$ is holomorphic, so $i\partial_{\alpha'}(I + \mathbb{H})f = \nabla_n(I + \mathbb{H})f$. Taking real parts gives the identity.

²⁴Here, to justify Green's identity, we can map (biholomorphically) the space P^- to the unit disk minus the slit, and then use the periodicity of all of the functions involved to consider the harmonic extensions of these functions to the whole unit disk.

where we have commuted $\partial_{\alpha'}$ outside the \mathbb{H} by (339) since $B|_{\partial} = 0$ (166).

We commute the $\frac{1}{Z_{,\alpha'}}$ factor inside the \mathbb{H} , and then apply the adjoint property (337):

$$\begin{aligned}
I_1 &= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \bar{\psi} d\alpha' + \Re \int i \left(\mathbb{H} \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \right) \Theta \bar{\psi} d\alpha' \\
&= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \bar{\psi} d\alpha' - \Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \mathbb{H} (\Theta \bar{\psi}) d\alpha' \\
&= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \Theta \bar{\psi} d\alpha' + \Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) [\bar{\psi}, \mathbb{H}] \Theta d\alpha' - \Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \bar{\psi} \mathbb{H} \Theta d\alpha' \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{173}$$

Observe that because $\mathbb{H}\Theta = \Theta$,

$$I_{13} = -\Re \int i \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} B \right) \bar{\psi} \Theta d\alpha' = -\Re \int i (\partial_{\alpha'} B) |\bar{\psi}|^2 d\alpha' = 0, \tag{174}$$

since $B \in \mathbb{R}$. It remains to control I_{11} and I_{12} .

We use Cauchy-Schwarz and then the $\dot{H}^{1/2} \times L^2$ commutator estimate (403) to control I_{12} :

$$|I_{12}| \leq \|D_{\alpha'} B\|_{L^2} \|[\bar{\psi}, \mathbb{H}] \Theta\|_{L^2} \lesssim \|D_{\alpha'} B\|_{L^2} \|\bar{\psi}\|_{\dot{H}^{1/2}} \|\Theta\|_{L^2}. \tag{175}$$

We have controlled $\|\bar{\psi}\|_{\dot{H}^{1/2}}$ and $\|\Theta\|_{L^2}$ at (168), and we will control $\|D_{\alpha'} B\|_{L^2}$ by (186) in Step 3 below.

It remains to control I_{11} from (173). Here we use Proposition 5, identity (340). Because $(I - \mathbb{H}) \frac{1}{Z_{,\alpha'}} = f \frac{1}{Z_{,\alpha'}}$ (362) and $f \partial_{\alpha'} B = 0$ by (166), we can rewrite

$$\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B = \mathbb{P}_H \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) - \frac{1}{2} f D_{\alpha'} B. \tag{176}$$

We plug (176) into I_{11} , and then use adjoint property (337):

$$\begin{aligned}
I_{11} &= \Re \int i \left\{ \mathbb{P}_H \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \right\} \Theta \bar{\psi} d\alpha' - \Re \left\{ \left(\frac{1}{2} f D_{\alpha'} B \right) \int i \Theta \bar{\psi} d\alpha' \right\} \\
&= \Re \int i \left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right) \mathbb{P}_A (\Theta \bar{\psi}) d\alpha' - \Re \left\{ \left(\frac{1}{2} f D_{\alpha'} B \right) \int i \Theta \bar{\psi} d\alpha' \right\}.
\end{aligned} \tag{177}$$

To control the first term, we use Cauchy-Schwarz, and then control the first factor with the $L^2 \times L^\infty$ estimate (394) and control the second factor by rewriting it as a commutator by (167) and then using the $\dot{H}^{1/2} \times L^2$ estimate (403). We use (49) to rewrite $\frac{1}{Z_{,\alpha'}} = -i \frac{Z_{tt} + i}{A_1}$ in the second term.

$$\begin{aligned}
|I_{11}| &\leq \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right\|_{L^2} \|\mathbb{P}_A (\Theta \bar{\psi})\|_{L^2} + \left| \left(\frac{1}{2} f D_{\alpha'} B \right) \int i \Theta \bar{\psi} d\alpha' \right| \\
&\leq \left\| \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} B \right\|_{L^2} \left\| \frac{1}{2} [\bar{\psi}, \mathbb{H}] \Theta \right\|_{L^2} + \|D_{\alpha'} B\|_{L^2} \int \left| \frac{Z_{tt} + i}{A_1} |\Theta|^2 \right| d\alpha' \\
&\lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|B\|_{L^\infty} \|\bar{\psi}\|_{\dot{H}^{1/2}} \|\Theta\|_{L^2} + \|D_{\alpha'} B\|_{L^2} \|Z_{tt} + i\|_{L^\infty} E_{a,\theta}.
\end{aligned} \tag{178}$$

We have controlled all the quantities on the last line, except for $\|D_{\alpha'} B\|_{L^2}$.

Step 3. Controlling $\|D_{\alpha'} B\|_{L^2}$. We must control $\|D_{\alpha'} B\|_{L^2}$, where B is as defined by (165). By (125),

$$B = (h_t \circ h^{-1})_{\alpha'} - \Re D_{\alpha'} Z_t = \Re D_{\alpha'} Z_t + \Re \left\{ - \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] \bar{Z}_{t,\alpha'} + [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\}. \tag{179}$$

Therefore, noting that $|\partial_{\alpha'} \Re f| \leq |\partial_{\alpha'} f|$ and so $|D_{\alpha'} \Re f| \leq |D_{\alpha'} f|$,

$$\|D_{\alpha'} B\|_{L^2} \leq \|D_{\alpha'}^2 Z_t\|_{L^2} + \left\| D_{\alpha'} \left[\frac{1}{\bar{Z}_{,\alpha'}}, \mathbb{H} \right] \bar{Z}_{t,\alpha'} \right\|_{L^2} + \left\| D_{\alpha'} [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}. \quad (180)$$

We've controlled $\|D_{\alpha'}^2 Z_t\|_{L^2}$, so it suffices to focus on the second and third terms. In what follows, we work on $D_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g$ for general functions f and g satisfying $f|_{\partial} = g|_{\partial} = 0$. Once we have an appropriate estimate, we will apply it to $f = \frac{1}{\bar{Z}_{,\alpha'}}$, $g = \bar{Z}_t$ for the second term, and $f = Z_t$, $g = \frac{1}{\bar{Z}_{,\alpha'}}$ for the third term.

We know

$$\begin{aligned} D_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g &= \frac{1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} \frac{1}{2i} \int (f(\alpha') - f(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \partial_{\beta'} g(\beta') d\beta' \\ &= \frac{1}{\bar{Z}_{,\alpha'}} (\partial_{\alpha'} f) \mathbb{H} \partial_{\alpha'} g - \frac{1}{\bar{Z}_{,\alpha'}} \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \partial_{\beta'} g(\beta') d\beta'. \end{aligned} \quad (181)$$

Via the boundedness of the Hilbert transform, we control the first of these terms by $\|D_{\alpha'} f\|_{L^\infty} \|\partial_{\alpha'} g\|_{L^2}$. Therefore, it suffices to focus on the second term. We commute the $\frac{1}{\bar{Z}_{,\alpha'}}$ inside, getting

$$- \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} D_{\beta'} g(\beta') d\beta' - \frac{\pi}{4i} \int \frac{(f(\alpha') - f(\beta')) \left(\frac{1}{\bar{Z}_{,\alpha'}}(\alpha') - \frac{1}{\bar{Z}_{,\beta'}}(\beta') \right)}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \partial_{\beta'} g(\beta') d\beta'. \quad (182)$$

We control the first term by (397):

$$\left\| \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} D_{\beta'} g(\beta') d\beta' \right\|_{L^2} \lesssim \|\partial_{\alpha'} f\|_{L^2} \|D_{\beta'} g\|_{L^\infty}. \quad (183)$$

We control the second term by (404):

$$\left\| \frac{\pi}{4i} \int \frac{(f(\alpha') - f(\beta')) \left(\frac{1}{\bar{Z}_{,\alpha'}}(\alpha') - \frac{1}{\bar{Z}_{,\beta'}}(\beta') \right)}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} \partial_{\beta'} g(\beta') d\beta' \right\|_{L^2} \lesssim \|\partial_{\alpha'} f\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \|\partial_{\alpha'} g\|_{L^2}. \quad (184)$$

We conclude that

$$\|D_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g\|_{L^2} \lesssim \|D_{\alpha'} f\|_{L^\infty} \|\partial_{\alpha'} g\|_{L^2} + \|D_{\alpha'} g\|_{L^\infty} \|\partial_{\alpha'} f\|_{L^2} + \|\partial_{\alpha'} f\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \|\partial_{\alpha'} g\|_{L^2}. \quad (185)$$

We can conclude from (180) and (185) that

$$\|D_{\alpha'} B\|_{L^2} \lesssim \|D_{\alpha'}^2 Z_t\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2} \left(\left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}^2 \right). \quad (186)$$

Step 4. Conclusion. We now combine our various estimates. We have

$$\begin{aligned} \left| \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \right| &\leq |I| + |II| \leq |I_1| + |I_2| + (164) \\ &\leq |I_{11}| + |I_{12}| + (171) + (164) \\ &\leq (178) + (175) + (171) + (164), \end{aligned} \quad (187)$$

where we use (186) to control $\|D_{\alpha'} B\|_{L^2}$.

In particular, by specifying $\theta = D_{\alpha'}^2 \bar{z}_t$, we have

$$\left| \Re \int i \left(\frac{h_\alpha^2}{|z_\alpha|^2} \right)_t \theta_\alpha \bar{\theta} d\alpha \right| \lesssim C(E). \quad (188)$$

8 Controlling G_θ of E_b

By (88), we must control

$$\left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha]\bar{z}_t|^2 d\alpha \right)^{1/2}. \quad (189)$$

We control the commutator via (417):

$$\left(\int \frac{1}{\mathfrak{a}} |[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha]\bar{z}_t|^2 d\alpha \right)^{1/2} \lesssim (\|D_\alpha z_{tt}\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty}^2) \|D_\alpha \bar{z}_t\|_{L^2(\frac{1}{\mathfrak{a}} d\alpha)}, \quad (190)$$

where we have controlled all the quantities on the RHS in §5.1. We are left with the term $\left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha)|^2 d\alpha \right)^{1/2}$. Since $\mathfrak{a}|z_\alpha|^2 = (A_1 \circ h)h_\alpha$ (50), $A_1 \geq 1$ (47), and $(\mathfrak{a}_t \bar{z}_\alpha) \circ h^{-1} = \mathcal{A}_t \bar{Z}_{\alpha'}$ we have

$$\left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha)|^2 d\alpha \right)^{1/2} \leq \left(\int \frac{1}{h_\alpha} |\partial_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha)|^2 d\alpha \right)^{1/2} = \left(\int |\partial_{\alpha'}(-i\mathcal{A}_t \bar{Z}_{\alpha'})|^2 d\alpha' \right)^{1/2}, \quad (191)$$

where in the second step we changed to Riemannian coordinates. We write $-i\mathcal{A}_t \bar{Z}_{\alpha'}$ as $\frac{\mathcal{A}_t}{\mathcal{A}}(-i\mathcal{A}\bar{Z}_\alpha)$ and apply $\partial_{\alpha'}$. Since $\bar{Z}_{tt} - i = -i\mathcal{A}\bar{Z}_{,\alpha'}$ (34), we have

$$\partial_{\alpha'}(-i\mathcal{A}_t \bar{Z}_{,\alpha'}) = (-i\mathcal{A}\bar{Z}_{,\alpha'})\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right) + \frac{\mathcal{A}_t}{\mathcal{A}}\bar{Z}_{tt,\alpha'}. \quad (192)$$

Therefore,

$$\left(\int |\partial_{\alpha'}(-i\mathcal{A}_t \bar{Z}_{,\alpha'})|^2 d\alpha' \right)^{1/2} \leq \left(\int \left| \mathcal{A}\bar{Z}_{,\alpha'}\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right) \right|^2 d\alpha' \right)^{1/2} + \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \|\bar{Z}_{tt,\alpha'}\|_{L^2}. \quad (193)$$

We controlled the factors in the second term on the RHS in (97) and (114). We can therefore concentrate on the first term.

We seek a way of writing $\mathcal{A}\bar{Z}_{,\alpha'}\partial_{\alpha'}\left(\frac{\mathcal{A}_t}{\mathcal{A}}\right)$. We will go through a slightly convoluted derivation that enables us to express this in terms of a commutator structure; we have to take care that we can invert $(I - \mathbb{H})$ and we want to make sure that the advantageous weight $\mathcal{A}\bar{Z}_{,\alpha'}$ is placed appropriately.

Starting from (192), we replace the LHS by the derivative of the LHS of our quasilinear equation (35), and then apply $(I - \mathbb{H})$ to the equation. We get

$$(I - \mathbb{H}) \left\{ (-i\mathcal{A}\bar{Z}_{,\alpha'})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right\} = (I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) - (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\bar{Z}_{tt} \right\}. \quad (194)$$

We want to move the factor $\mathcal{A}\bar{Z}_{,\alpha'}$ outside on the LHS, so we do this:

$$(-i\mathcal{A}\bar{Z}_{,\alpha'})(I - \mathbb{H})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} = (I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) - (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\bar{Z}_{tt} \right\} + [i\mathcal{A}\bar{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}. \quad (195)$$

Now $\frac{\mathcal{A}_t}{\mathcal{A}}$ is real and \mathbb{H} is purely imaginary, so $|\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}| \leq |(I - \mathbb{H})\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}|$. Taking absolute value on both sides, we have

$$\left| \mathcal{A}\bar{Z}_{,\alpha'}\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right| \leq \left| (I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) - (I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\partial_{\alpha'}\bar{Z}_{tt} \right\} + [i\mathcal{A}\bar{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right|. \quad (196)$$

We can easily control the L^2 norm of the second and third terms. By the L^2 boundedness of \mathbb{H} and Hölder for the second term and estimate (394) for the third term, and since $i\mathcal{A}\bar{Z}_{,\alpha'} = -(\bar{Z}_{tt} - i)$,

$$\left\| -(I - \mathbb{H}) \left\{ \frac{\mathcal{A}_t}{\mathcal{A}}\bar{Z}_{tt,\alpha'} \right\} - [i\mathcal{A}\bar{Z}_{,\alpha'}, \mathbb{H}]\partial_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}. \quad (197)$$

We can now focus on controlling

$$\|(I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2} = \|\partial_{\alpha'}(I - \mathbb{H})(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2}, \quad (198)$$

where we used $(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})|_{\partial} = 0$ (352) to commute $\partial_{\alpha'}$ outside $(I - \mathbb{H})$ by (339).

Recall from (58) and $i\mathcal{A} = \frac{Z_{tt} + i}{Z_{,\alpha'}}$ that

$$\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'} = (D_{\alpha'}^2 \bar{Z}_t) Z_t^2 + 2Z_t D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2(D_{\alpha'} \bar{Z}_t) Z_{tt} + F_{tt} \circ Z + iD_{\alpha'} \bar{Z}_t, \quad (199)$$

where $F(z(\alpha, t), t) := \bar{z}_t(\alpha, t)$. Under $(I - \mathbb{H})$, by (369), (374) and (373) the last two terms disappear and the rest turn into commutators:

$$(I - \mathbb{H})(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) = [Z_t^2, \mathbb{H}]D_{\alpha'}^2 \bar{Z}_t + 2[Z_t, \mathbb{H}]D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2[Z_{tt}, \mathbb{H}]D_{\alpha'} \bar{Z}_t. \quad (200)$$

Therefore, by (198) and (200), we have to control the L^2 norm of

$$\partial_{\alpha'}[Z_t^2, \mathbb{H}]D_{\alpha'}^2 \bar{Z}_t + 2\partial_{\alpha'}[Z_t, \mathbb{H}]D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2\partial_{\alpha'}[Z_{tt}, \mathbb{H}]D_{\alpha'} \bar{Z}_t. \quad (201)$$

We use the identity

$$\partial_{\alpha'}[f, \mathbb{H}]g = f' \mathbb{H}g - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' \quad (202)$$

to expand out each term in (201), and use (369) and (374) to remove the \mathbb{H} s from the RHS. We get

$$\begin{aligned} (201) &= 2Z_t Z_{t,\alpha'} D_{\alpha'}^2 \bar{Z}_t + 2Z_{t,\alpha'} D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2Z_{tt,\alpha} D_{\alpha'} \bar{Z}_t - \frac{\pi}{4i} \int \frac{Z_t^2(\alpha') - Z_t^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'}^2 \bar{Z}_t d\beta' \\ &\quad - \frac{\pi}{2i} \int \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'}(\bar{Z}_{tt} - (D_{\beta'} \bar{Z}_t) Z_t) d\beta' - \frac{\pi}{2i} \int \frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \bar{Z}_t d\beta'. \end{aligned} \quad (203)$$

We expand out the RHS. We note that certain terms cancel out with others, and we further observe the following identity:

$$\int \frac{f^2(\alpha') - f^2(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' - 2 \int \frac{(f(\alpha') - f(\beta'))f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' = \int \frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta'. \quad (204)$$

We have

$$\begin{aligned} (201) &= 2Z_{t,\alpha'}(D_{\alpha'} \bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) D_{\alpha'} Z_t) + 2Z_{tt,\alpha} D_{\alpha'} \bar{Z}_t - \frac{\pi}{4i} \int \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'}^2 \bar{Z}_t d\beta' \\ &\quad - \frac{\pi}{2i} \int \frac{Z_t(\alpha') - Z_t(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (D_{\beta'} \bar{Z}_{tt} - (D_{\beta'} \bar{Z}_t) D_{\beta'} Z_t) d\beta' - \frac{\pi}{2i} \int \frac{Z_{tt}(\alpha') - Z_{tt}(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} D_{\beta'} \bar{Z}_t d\beta'. \end{aligned} \quad (205)$$

We now apply Hölder to the first two terms, (404) to the third term, and (397) to the last two terms. We get

$$\begin{aligned} \|(I - \mathbb{H})\partial_{\alpha'}(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2} &= \|(201)\|_{L^2} \\ &\lesssim \|Z_{t,\alpha'}\|_{L^2} (\|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}^2) + \|Z_{tt,\alpha'}\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} + \|Z_{t,\alpha'}\|_{L^2}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}. \end{aligned} \quad (206)$$

We now combine our various estimates. We have

$$\left\| \mathcal{A} \bar{Z}_{,\alpha'} \partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \leq (197) + (206) \lesssim C(E) \quad (207)$$

and

$$\begin{aligned} & \left(\int \frac{1}{\mathfrak{a}} |D_\alpha(-i\mathfrak{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha] \bar{z}_t|^2 d\alpha \right)^{1/2} \leq (190) + (193) \\ & \leq (190) + (207) + \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \|\bar{Z}_{tt, \alpha'}\|_{L^2} \lesssim C(E). \end{aligned} \quad (208)$$

We can now conclude that $\frac{d}{dt} E_b$ is bounded by a polynomial of E .
We record here the estimate

$$\left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \leq \left\| \mathcal{A} \bar{Z}_{, \alpha'} \partial_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim C(E), \quad (209)$$

which holds because $|\mathcal{A} \bar{Z}_{, \alpha'}| = \frac{A_1}{|Z_{, \alpha'}|} \geq \frac{1}{|Z_{, \alpha'}|}$; we will use this in §9.

9 Controlling G_θ of E_a

From (86), we must control

$$\left(\int |D_\alpha^2(-i\mathfrak{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^2] \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}. \quad (210)$$

Recall that $A_1 \geq 1$ (47). We control the commutator via (418) and Hölder:

$$\begin{aligned} & \|[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^2] \bar{z}_t\|_{L^2(\frac{h_\alpha}{A_1 \circ h} d\alpha)} \leq \|[\partial_t^2 + i\mathfrak{a}\partial_\alpha, D_\alpha^2] \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ & \lesssim \|D_\alpha z_{tt}\|_{L^\infty} \|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 \bar{z}_t\|_{L^2(h_\alpha d\alpha)} + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)} \\ & + \|D_\alpha^2 z_{tt}\|_{L^2(h_\alpha d\alpha)} \|D_\alpha \bar{z}_t\|_{L^\infty} + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 z_t\|_{L^2(h_\alpha d\alpha)} + \|D_\alpha^2 z_t\|_{L^2(h_\alpha d\alpha)} \|D_\alpha \bar{z}_{tt}\|_{L^\infty} \\ & + \|D_\alpha z_t\|_{L^\infty} \|D_\alpha^2 \bar{z}_{tt}\|_{L^2(h_\alpha d\alpha)}. \end{aligned} \quad (211)$$

We have controlled all quantities on the RHS in §5.1. We are left with the term $\left(\int |D_\alpha^2(\mathfrak{a}_t \bar{z}_\alpha)|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2}$.

We know

$$\left(\int |D_\alpha^2(\mathfrak{a}_t \bar{z}_\alpha)|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \right)^{1/2} \leq \left(\int |D_\alpha^2(\mathfrak{a}_t \bar{z}_\alpha)|^2 h_\alpha d\alpha \right)^{1/2} = \left(\int |D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_\alpha)|^2 d\alpha' \right)^{1/2}, \quad (212)$$

where we changed to Riemannian coordinate in the second step. We will now focus on estimating

$$\left(\int |D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_\alpha)|^2 d\alpha' \right)^{1/2}. \quad (213)$$

Our plan will be to rewrite $D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_\alpha)$ in a form which we can control via our energy. We will take advantage of the fact that $\frac{\mathcal{A}_t}{\mathcal{A}}$ is real-valued and use $(I - \mathbb{H})$ to turn the main part of $D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_\alpha)$ into a series of commutators, as we did in §3.2.3. Getting to this point will require a long and somewhat convoluted series of calculations.

We begin by writing $\mathcal{A}_t \bar{Z}_\alpha = (\frac{\mathcal{A}_t}{\mathcal{A}}) \mathcal{A} \bar{Z}_\alpha$. By the product rule

$$D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_\alpha) = \left(D_{\alpha'}^2 \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) \right) \mathcal{A} \bar{Z}_\alpha + 2D_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) D_{\alpha'}(\mathcal{A} \bar{Z}_\alpha) + \frac{\mathcal{A}_t}{\mathcal{A}} D_{\alpha'}^2(\mathcal{A} \bar{Z}_\alpha). \quad (214)$$

We can handle the second and third terms directly, using $\mathcal{A} \bar{Z}_\alpha = i(\bar{Z}_{tt} - i)$ (34):

$$\left\| 2D_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) D_{\alpha'}(\mathcal{A} \bar{Z}_\alpha) + \frac{\mathcal{A}_t}{\mathcal{A}} D_{\alpha'}^2(\mathcal{A} \bar{Z}_\alpha) \right\|_{L^2} \leq 2 \left\| D_{\alpha'} \left(\frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^\infty} \|D_\alpha \bar{Z}_{tt}\|_{L^\infty} + \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \|D_\alpha^2 \bar{Z}_{tt}\|_{L^2}, \quad (215)$$

where we have controlled all the quantities on the RHS in §5.1 and in (114) and (209). It therefore suffices to focus on the first term on the RHS of (214), $(D_{\alpha'}^2(\frac{\mathcal{A}_t}{\mathcal{A}}))\mathcal{A}\bar{Z}_\alpha = i(\bar{Z}_{tt} - i)D_{\alpha'}^2(\frac{\mathcal{A}_t}{\mathcal{A}})$.

Step 1. *Controlling $(D_{\alpha'}^2(\frac{\mathcal{A}_t}{\mathcal{A}}))\mathcal{A}\bar{Z}_\alpha$ or $i(\bar{Z}_{tt} - i)D_{\alpha'}^2(\frac{\mathcal{A}_t}{\mathcal{A}})$.* We now rearrange this term so that we can apply $(I - \mathbb{H})$ in a way so that we will be able to invert the operator by taking real parts. Note that $\frac{\mathcal{A}_t}{\mathcal{A}}$ is purely real. Unfortunately, our derivative $D_{\alpha'} = \frac{1}{\bar{Z}_{,\alpha'}}\partial_{\alpha'}$ is not purely real. To get around this, we factor the derivative into a real derivative and a complex modulus-one weight. Recall our notation $|D_{\alpha'}| = \frac{1}{|\bar{Z}_{,\alpha'}|}\partial_{\alpha'}$.

Since $D_{\alpha'} = \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)|D_{\alpha'}|$, we rewrite

$$D_{\alpha'}^2 = \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 |D_{\alpha'}|^2 + \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)\right) |D_{\alpha'}|. \quad (216)$$

Therefore,

$$i(\bar{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} = i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 |D_{\alpha'}|^2 \frac{\mathcal{A}_t}{\mathcal{A}} + i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)\right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}. \quad (217)$$

It will turn out that we'll want to replace $|D_{\alpha'}|^2$ in the first term of the RHS of (217) with $\partial_{\alpha'} \left(\frac{1}{|\bar{Z}_{,\alpha'}|} |D_{\alpha'}|\right)$, since this will give us the proper commutator estimate,²⁵ and it further turns out that we want to make the switch now rather than later, when it would be in commutator form. Doing this, we get

$$\begin{aligned} i(\bar{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} &= i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 \partial_{\alpha'} \left(\frac{1}{|\bar{Z}_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}\right) \\ &\quad - \underbrace{i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 \left(\partial_{\alpha'} \frac{1}{|\bar{Z}_{,\alpha'}|}\right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} + i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)\right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}}_e, \end{aligned} \quad (218)$$

where we will use

$$e := -i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 \left(\partial_{\alpha'} \frac{1}{|\bar{Z}_{,\alpha'}|}\right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} + i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right) \left(|D_{\alpha'}| \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)\right) |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \quad (219)$$

as an abbreviation to save space and de-emphasize the less central error terms, which we will control directly, below at (224). We now apply $(I - \mathbb{H})$ to both sides of (218):

$$(I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} = (I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 \partial_{\alpha'} \left(\frac{1}{|\bar{Z}_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}\right) \right\} + (I - \mathbb{H})e. \quad (220)$$

Observe that the first term on the RHS is purely real, except for the controllable factor $-i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2$.

We commute that part outside the $(I - \mathbb{H})$. We get

$$\begin{aligned} (I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} &= i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2 (I - \mathbb{H}) \partial_{\alpha'} \left(\frac{1}{|\bar{Z}_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}\right) \\ &\quad + \left[i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{\bar{Z}_{,\alpha'}}\right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|\bar{Z}_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}\right) + (I - \mathbb{H})e. \end{aligned} \quad (221)$$

²⁵See footnote 26 below for a further explanation.

Because \mathbb{H} is purely imaginary, $\left| \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right| \leq \left| (I - \mathbb{H}) \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right|$. By taking absolute values, we have

$$\begin{aligned} & \left| i(\bar{Z}_{tt} - i) \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right| \\ & \leq \left| (I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} - \left[i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) - (I - \mathbb{H})e \right|. \end{aligned} \quad (222)$$

Now we may begin controlling these terms. Recall that what we needed to control was the L^2 norm of (218). We can estimate this by

$$\begin{aligned} \left\| -i(\bar{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} & \lesssim \left\| i(\bar{Z}_{tt} - i) \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} + \|e\|_{L^2} \lesssim \|(222)\|_{L^2} + \|e\|_{L^2} \\ & \lesssim \left\| (I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i) D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\} \right\|_{L^2} \\ & \quad + \left\| \left[i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} + \|e\|_{L^2}. \end{aligned} \quad (223)$$

Thus, it suffices to focus on these three terms. The first term will be the main term that will eventually give us the commutator structure that we want. We first control the remaining terms.

First we check the error term, e (219). We control

$$\begin{aligned} \|e\|_{L^2} & \leq \left(\left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_{L^\infty} + \left\| (\bar{Z}_{tt} - i) \left(|D_{\alpha'}| \frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right) \right\|_{L^\infty} \right) \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \\ & \lesssim \left\| (\bar{Z}_{tt} - i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2}, \end{aligned} \quad (224)$$

where in the second step we used $|\partial_{\alpha'}|f| \leq |\partial_{\alpha'} f|$ (391), and $\left| \partial_{\alpha'} \frac{f}{|f|} \right| \leq \left| \frac{f'}{|f|} \right|$ (391). We have controlled both factors on the RHS in (142) and (209).

Now we estimate the second term on the RHS of (223). We use $L^2 \times L^\infty$ commutator estimate (394). Observe that because either (i) the angle $\nu = \frac{\pi}{2}$, in which case $\left. \frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right|_{\partial} = 0$, or else (ii) the angle $\nu \neq \frac{\pi}{2}$ and so $\bar{Z}_{tt} - i = 0$ at the corners (see (351)) this commutator does satisfy the boundary condition we need for (394).²⁶ We have

$$\begin{aligned} & \left\| \left[-i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} \\ & \lesssim \left\| \partial_{\alpha'} \left(-i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \right) \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \lesssim \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}, \end{aligned} \quad (226)$$

²⁶ Note that this estimate for $[f, \mathbb{H}] \partial_{\alpha'} g$, unlike the $L^\infty \times L^2$ estimate, does *not* require that $g|_{\partial} = 0$. This explains why we moved from $|D_{\alpha'}|^2 \frac{\mathcal{A}_t}{\mathcal{A}}$ to $\partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right)$; had we tried to estimate

$$\left\| \left[\frac{1}{|Z_{,\alpha'}|} \left(-i(\bar{Z}_{tt} - i) \left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}} \right)^2 \right), \mathbb{H} \right] \partial_{\alpha'} |D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \quad (225)$$

by $L^\infty \times L^2$ commutator estimate (398), we would have been able to control each of the terms (they would be similar to the estimates for e in (224) above), but we don't know that $|D_{\alpha'}| \frac{\mathcal{A}_t}{\mathcal{A}}|_{\partial}$ is zero.

where in the second step we replaced $\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}$ with $\frac{\bar{Z}_{tt}-i}{|\bar{Z}_{tt}-i|}$ (392) to obtain the last inequality. We have controlled the first factor, $\|Z_{tt,\alpha'}\|_{L^2}$, in §5.1. What remains to be done is to show that we can control $\left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}\frac{\mathcal{A}_t}{\mathcal{A}}\right\|_{L^\infty}$. We defer doing so until the end, in Step 7.

We're left with the first, main term of the RHS of (223). Observe that by (214), our main equation $\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'} = -i\mathcal{A}_t\bar{Z}_{,\alpha'}$ (35), and the L^2 boundedness of \mathbb{H} ,

$$\begin{aligned} \left\|(I - \mathbb{H}) \left\{ i(\bar{Z}_{tt} - i)D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\}\right\|_{L^2} &\lesssim \|(I - \mathbb{H}) (D_{\alpha'}^2(\mathcal{A}_t\bar{Z}_{,\alpha'}))\|_{L^2} + (215) \\ &= \|(I - \mathbb{H})D_{\alpha'}^2(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2} + (215). \end{aligned} \quad (227)$$

We have therefore reduced things (except for the one term we're deferring to the end) to controlling $\|(I - \mathbb{H})D_{\alpha'}^2(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})\|_{L^2}$, which will allow us to use desirable commutators.

Step 2. *Controlling* $\|(I - \mathbb{H}) (D_{\alpha'}^2(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}))\|_{L^2}$. We now expand out $(I - \mathbb{H})D_{\alpha'}^2(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'})$, as we did in §3.2.3. From (58), we have

$$\bar{Z}_{ttt} = (D_{\alpha'}^2\bar{Z}_t)Z_t^2 + 2Z_tD_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'}\bar{Z}_t)Z_t) + (D_{\alpha'}\bar{Z}_t)Z_{tt} + F_{tt} \circ Z, \quad (228)$$

where $F(z(\alpha, t), t) := \bar{z}_t(\alpha, t)$. By $Z_{tt} + i = i\mathcal{A}Z_{,\alpha'}$ (34), $i\mathcal{A}\bar{Z}_{t,\alpha'} = (Z_{tt} + i)D_{\alpha'}\bar{Z}_t$. Putting these together and applying $(I - \mathbb{H})D_{\alpha'}^2$, we have

$$\begin{aligned} (I - \mathbb{H})D_{\alpha'}^2(\bar{Z}_{ttt} + i\mathcal{A}\bar{Z}_{t,\alpha'}) \\ = (I - \mathbb{H})D_{\alpha'}^2 \left((D_{\alpha'}^2\bar{Z}_t)Z_t^2 + 2Z_tD_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'}\bar{Z}_t)Z_t) + (D_{\alpha'}\bar{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z \right). \end{aligned} \quad (229)$$

Observe that each of these terms has holomorphic factors, which will (hopefully) give us controllable commutators or disappear under $(I - \mathbb{H})$.

First we address the last two terms, $(I - \mathbb{H})D_{\alpha'}^2 \left((D_{\alpha'}\bar{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z \right)$. We may rewrite

$$\begin{aligned} (I - \mathbb{H})D_{\alpha'}^2 \left((D_{\alpha'}\bar{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z \right) \\ = 2(I - \mathbb{H}) \left((D_{\alpha'}^3\bar{Z}_t)(Z_{tt} + i) + 2(D_{\alpha'}^2\bar{Z}_t)(D_{\alpha'}Z_{tt}) + (D_{\alpha'}\bar{Z}_t)(D_{\alpha'}^2Z_{tt}) \right), \end{aligned} \quad (230)$$

where we have used (373) and (369).

By Hölder, we can control the second and third terms of the RHS of (230):

$$\begin{aligned} \|(I - \mathbb{H}) \left(2(D_{\alpha'}^2\bar{Z}_t)(D_{\alpha'}Z_{tt}) + (D_{\alpha'}\bar{Z}_t)(D_{\alpha'}^2Z_{tt}) \right)\|_{L^2} \\ \lesssim \|D_{\alpha'}^2\bar{Z}_t\|_{L^2} \|D_{\alpha'}Z_{tt}\|_{L^\infty} + \|D_{\alpha'}\bar{Z}_t\|_{L^\infty} \|D_{\alpha'}^2Z_{tt}\|_{L^2}. \end{aligned} \quad (231)$$

To control the first term on the RHS of (230), we use (370) to write

$$\begin{aligned} \|(I - \mathbb{H})(D_{\alpha'}^3\bar{Z}_t)(Z_{tt} + i)\|_{L^2} &= \left\| \left[\frac{Z_{tt} + i}{Z_{,\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \\ &\lesssim \left\| \partial_{\alpha'} \frac{Z_{tt} + i}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \\ &\lesssim \left(\|D_{\alpha'}Z_{tt}\|_{L^\infty} + \left\| (Z_{tt} + i) \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \right) \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \end{aligned} \quad (232)$$

where we have used the $L^\infty \times L^2$ commutator estimate (398).

We conclude from (230), (231) and (232) that we can control

$$\|(I - \mathbb{H})D_{\alpha'}^2 \left((D_{\alpha'}\bar{Z}_t)(2Z_{tt} + i) + F_{tt} \circ Z \right)\|_{L^2} \lesssim (231) + (232) \lesssim C(E). \quad (233)$$

We can now focus on what remains from (229),

$$\begin{aligned}
& (I - \mathbb{H}) D_{\alpha'}^2 \{ (D_{\alpha'}^2 \bar{Z}_t) Z_t^2 + 2 Z_t D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \} \\
&= (I - \mathbb{H}) \{ (D_{\alpha'}^2 (Z_t^2)) (D_{\alpha'}^2 \bar{Z}_t) + 2 (D_{\alpha'}^2 Z_t) D_{\alpha'} (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \} \\
&+ (I - \mathbb{H}) \{ 2 (D_{\alpha'} (Z_t^2)) (D_{\alpha'}^3 \bar{Z}_t) + 4 (D_{\alpha'} Z_t) D_{\alpha'}^2 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \} \\
&+ (I - \mathbb{H}) \{ Z_t^2 (D_{\alpha'}^4 \bar{Z}_t) + 2 Z_t D_{\alpha'}^3 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \} \\
&= i + ii + iii.
\end{aligned} \tag{234}$$

Terms i and ii are easy to control. The details are uninteresting and distract from our main derivation, so we defer them to Step 6, below.

We may therefore focus on iii .

Step 3. *Controlling iii from (234).* We now focus on controlling

$$iii = (I - \mathbb{H}) Z_t^2 (D_{\alpha'}^4 \bar{Z}_t) + (I - \mathbb{H}) 2 Z_t D_{\alpha'}^3 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) = I + II. \tag{235}$$

These terms appear to require too much regularity to close the estimate: we have up to four derivatives here, when our energy controls only two derivatives. Our strategy is to expand the equation and then exploit the holomorphicity of certain of the factors involved; other terms can be controlled directly, and the rest of the terms cancel out with each other.

We begin by expanding II from (235). Rewriting the identity as $\mathbb{P}_A + \mathbb{P}_H$, we have

$$\begin{aligned}
II &= 2(I - \mathbb{H}) \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \right\} \\
&= 2 \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + \underbrace{(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \right\}}_{0 \text{ by (376)}},
\end{aligned} \tag{236}$$

where we've used (375) to get the first commutator. We now expand what remains from (236):

$$\begin{aligned}
II &= 2 \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} (D_{\alpha'}^2 \bar{Z}_{tt} - (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} - 2(D_{\alpha'}^2 \bar{Z}_t) D_{\alpha'} Z_t - (D_{\alpha'} \bar{Z}_t) D_{\alpha'}^2 Z_t) \\
&= II_1 + II_2 + II_3 + II_4.
\end{aligned} \tag{237}$$

We control everything but II_2 directly by the $L^\infty \times L^2$ commutator estimate (398). To apply this, we need both parts to have periodic boundary behavior. We know $\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial} = 0$ (356). The reason that

$$(D_{\alpha'}^2 \bar{Z}_{tt} - 2(D_{\alpha'}^2 \bar{Z}_t) D_{\alpha'} Z_t - (D_{\alpha'} \bar{Z}_t) D_{\alpha'}^2 Z_t) \Big|_{\partial} = 0 \tag{238}$$

is that it equals $D_{\alpha'} (F_{tz} \circ Z) + (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}}$, both of which have periodic boundary behavior by (346), (347), and (359). We get

$$\begin{aligned}
& \|II_1 + II_3 + II_4\|_{L^2} \\
&\lesssim \left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} (\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \|D_{\alpha'} Z_t\|_{L^\infty} + \|D_{\alpha'}^2 Z_t\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}).
\end{aligned} \tag{239}$$

We are now left with II_2 from (237). We switch back from the commutator form, commuting the $\partial_{\alpha'}$ in the second term outside $(I - \mathbb{H})$, which we can do because $(\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial} = 0$ (346) (347), and then we

expand the factor $\frac{Z_t}{Z_{,\alpha'}}$ by rewriting the identity as $\mathbb{P}_A + \mathbb{P}_H$:

$$\begin{aligned}
II_2 &= -2(I - \mathbb{H}) \left\{ \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} \left((\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} \right) \right\} + 2\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} (I - \mathbb{H}) \left((\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} \right) \\
&= -2(I - \mathbb{H}) \left\{ \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} \left((\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\} \\
&\quad - 2(I - \mathbb{H}) \left\{ \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} \left((\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \right\} + 2\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\} \\
&= II_{21} + II_{22} + II_{23},
\end{aligned} \tag{240}$$

where in the penultimate line we used $(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right\} = 0$ by (378).

Next, we expand I from (235):

$$I = (I - \mathbb{H}) \frac{Z_t^2}{Z_{,\alpha'}^2} \partial_{\alpha'}^2 D_{\alpha'}^2 \bar{Z}_t + (I - \mathbb{H}) Z_t^2 \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t = I_1 + I_2. \tag{241}$$

We once again expand the identity operator into the sum of the projections $\mathbb{P}_A + \mathbb{P}_H$:

$$\begin{aligned}
I_1 &= (I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \bar{Z}_t \right\} + (I - \mathbb{H}) \left\{ 2 \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'}^2 D_{\alpha'}^2 \bar{Z}_t \right\} \\
&\quad + (I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'}^2 D_{\alpha'}^2 \bar{Z}_t \right\} \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{242}$$

We now carefully inspect (240), (241) and (242) and combine terms.

First, we combine II_{21} , I_{11} , and I_{23} , using (370) to rewrite II_{23} in commutator form:

$$II_{21} + I_{11} + II_{23} = -(I - \mathbb{H}) \partial_{\alpha'} \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} + 2\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} \left[\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}, \mathbb{H} \right] (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t). \tag{243}$$

Commuting the $\partial_{\alpha'}$ in the first term outside $(I - \mathbb{H})$, then rewriting it in commutator form by (370), we have

$$\begin{aligned}
II_{21} + I_{11} + II_{23} &= -\partial_{\alpha'} \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2, \mathbb{H} \right] \{ \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \} + 2\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \partial_{\alpha'} \left[\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}, \mathbb{H} \right] (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \\
&\quad - [\partial_{\alpha'}, \mathbb{H}] \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \\
&= -\frac{1}{2i} \int \frac{\pi}{2} \frac{\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}(\alpha') - \mathbb{P}_A \frac{Z_t}{Z_{,\beta'}}(\beta') \right)^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \partial_{\beta'} D_{\beta'}^2 \bar{Z}_t d\beta' - [\partial_{\alpha'}, \mathbb{H}] \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right)^2 \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \\
&=: - \left[\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}, \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}; \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right] + R_1,
\end{aligned} \tag{244}$$

where we arrived at the second equality by applying the derivatives and combining the first two terms. We can control the first term on the RHS via (410), relying on boundary condition (356) (and (347)):

$$\left\| \left[\mathbb{P}_A \frac{Z_t}{Z_{,\alpha}}, \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}}; \partial_{\beta} D_{\beta'}^2 \bar{Z}_t \right] \right\|_{L^2} \leq \left\| \partial_{\alpha'} \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\|_{L^\infty}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \tag{245}$$

where we have controlled the first factor at (147) and the second factor in §5.1. We will handle the remainder R_1 a little later, after we combine the next three terms in II_2 and I .

We next combine II_{22} , I_2 and I_{12} . We note that after applying the derivative in II_{22} by the product rule, one of the terms cancels out with I_{12} . We have

$$II_{22} + I_2 + I_{12} = (I - \mathbb{H}) \left\{ -2 \left(\mathbb{P}_A \frac{Z_t}{Z_{\alpha'}} \right) \left((\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{\alpha'}} \right) + Z_t^2 \left(\frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\}, \quad (246)$$

where we've used $\frac{Z_t}{Z_{\alpha'}} \Big|_{\partial} = 0$ (355) to commute the $\partial_{\alpha'}$ in the first term inside \mathbb{P}_H by (339). We rewrite

$$\begin{aligned} II_{22} + I_2 + I_{12} &= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left\{ \frac{Z_t}{Z_{\alpha'}} Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - 2 \left(\mathbb{P}_A \frac{Z_t}{Z_{\alpha'}} \right) \left(\mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{\alpha'}} \right) \right\} \right\} \\ &= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left\{ \frac{Z_t}{Z_{\alpha'}} Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - 2 \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{Z_{\alpha'}} \right) \left(\mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{\alpha'}} \right) \right\} \right\} \\ &\quad + 2(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{\alpha'}} \right) \left(\partial_{\alpha'} \mathbb{P}_H \frac{Z_t}{Z_{\alpha'}} \right) \right\} \right\} \\ &=: A + R_2. \end{aligned} \quad (247)$$

We begin by considering the second term, R_2 , which we claim will cancel out with I_{13} from (242) and R_1 from (244). Indeed, these three terms are

$$\begin{aligned} R_1 + I_{13} + R_2 &= -[\partial_{\alpha'}, \mathbb{H}] \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{\alpha'}} \right)^2 \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} + (I - \mathbb{H}) \partial_{\alpha'} \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_H \frac{Z_t}{Z_{\alpha'}} \right)^2 \right\} \\ &= -[\partial_{\alpha'}, \mathbb{H}] \left\{ \left(\mathbb{P}_A \frac{Z_t}{Z_{\alpha'}} \right)^2 \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} + \underbrace{\partial_{\alpha'} (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_H \frac{Z_t}{Z_{\alpha'}} \right)^2 \right\}}_{0 \text{ by (378)}} \\ &\quad + [\partial_{\alpha'}, \mathbb{H}] \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_H \frac{Z_t}{Z_{\alpha'}} \right)^2 \right\} \\ &= -[\partial_{\alpha'}, \mathbb{H}] \left\{ \left((\mathbb{P}_A - \mathbb{P}_H) \frac{Z_t}{Z_{\alpha'}} \right) \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{Z_{\alpha'}} \right) (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \right\} \\ &= [\partial_{\alpha'}, \mathbb{H}] \left\{ \left(\mathbb{H} \frac{Z_t}{Z_{\alpha'}} \right) \left(\frac{Z_t}{Z_{\alpha'}} \right) (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \right\} = 0. \end{aligned} \quad (248)$$

This is zero because the first factor has periodic boundary behavior by (356), the Z_t has periodic boundary behavior by (346), and the $\frac{1}{Z_{\alpha'}}$ joins with the $\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t$ to ensure periodic boundary behavior by (347).

Therefore

$$iii = I + II = - \left[\mathbb{P}_A \frac{Z_t}{Z_{\alpha'}}, \mathbb{P}_A \frac{Z_t}{Z_{\alpha'}}; \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right] + A + (II_1 + II_3 + II_4). \quad (249)$$

Since we have estimated the first and third sets of terms at (245) and (239) above, all that remains is to control

$$A = (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{\alpha'}} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - 2 \mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{\alpha'}} \right) \right\}. \quad (250)$$

Let $(*) = Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - 2 \mathbb{P}_H \partial_{\alpha'} \frac{Z_t}{Z_{\alpha'}}$. We expand $(*)$ as follows:

$$\begin{aligned} (*) &= Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - (I + \mathbb{H}) D_{\alpha'} Z_t - (I + \mathbb{H}) \left\{ Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\} \\ &= -\mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) - (I + \mathbb{H}) D_{\alpha'} Z_t. \end{aligned} \quad (251)$$

Therefore, we need to control

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \quad (252)$$

and

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} (I + \mathbb{H}) D_{\alpha'} Z_t \right\}. \quad (253)$$

We will control (252) in the next step, Step 4, and (253) in Step 5.

Step 4. Controlling (252). We begin with (252). Our goal is to take advantage of the fact that we can control $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}$. Therefore, we rewrite (252) so that we can isolate $\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t$ for a commutator estimate:

$$\begin{aligned} (252) &= (I - \mathbb{H}) \left\{ \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right) Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \\ &= (I - \mathbb{H}) \left\{ \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\} \\ &\quad - (I - \mathbb{H}) \left\{ \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'}^2 \bar{Z}_t \right\} \\ &= (a) + (b). \end{aligned} \quad (254)$$

We expand (a) using $I = \mathbb{P}_A + \mathbb{P}_H$:

$$\begin{aligned} (a) &= (I - \mathbb{H}) \left\{ \left((\mathbb{P}_A + \mathbb{P}_H) \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\} \\ &= (I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\} \\ &\quad + \underbrace{(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\}}_{0 \text{ by (380)}} \\ &= \left[\left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right), \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \end{aligned} \quad (255)$$

by (379) and (380). We now use (399) to estimate the RHS of (255), using the $\dot{H}^{1/2}$ control:

$$\|(a)\|_{L^2} \lesssim \left\| \partial_{\alpha'} \mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}. \quad (256)$$

To apply (399), we require

$$\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \Big|_{\partial} = 0. \quad (257)$$

(Note that the second required boundary condition, $\left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \Big|_{\partial} = 0$, holds by (347) and (350).) It isn't obvious why this first boundary condition should hold. We will show below at (268) that it does, in fact, hold. Assuming that, because we've controlled $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}$ at (91), we've reduced things to controlling $\left\| \partial_{\alpha'} \left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \right\|_{L^2}$.

We first find a useful way of rewriting $Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)$. Observe that

$$[Z_t, [Z_t, \mathbb{H}]] = Z_t^2 \mathbb{H} - 2Z_t \mathbb{H} Z_t + \mathbb{H} Z_t^2. \quad (258)$$

Therefore,

$$\begin{aligned} [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} &= Z_t^2 \mathbb{H} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} - 2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) + \mathbb{H} \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \\ &= (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - 2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right), \end{aligned} \quad (259)$$

where $\mathbb{H} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}$ by (368). Therefore,

$$2Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) = (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}. \quad (260)$$

Now we apply \mathbb{P}_A to this, getting

$$2\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) = \mathbb{P}_A (I + \mathbb{H}) \left(Z_t^2 \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - \mathbb{P}_A [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}. \quad (261)$$

By (336), the first term on the RHS is just a mean term. Because this mean will eventually disappear under the derivative, we denote it simply by c . We are left with

$$2\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) = c - \mathbb{P}_A [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}. \quad (262)$$

Now we will show, as promised, that $\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \Big|_{\partial} = 0$. We expand

$$[Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} = \frac{1}{2i} \int (Z_t(\alpha') - Z_t(\beta'))^2 \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) \partial_{\beta'} \frac{1}{Z_{,\beta'}} d\beta'. \quad (263)$$

Because $Z_t|_{\partial} = 0$, we see, by the fundamental theorem of calculus and Hölder, that the kernel function $(Z_t(\alpha') - Z_t(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta'))$ is continuous for $(\alpha', \beta') \in I \times I \setminus \{|\alpha' - \beta'| \notin \{0, 2\}\}$, periodic in α' , and satisfies

$$|(Z_t(\alpha') - Z_t(\beta'))^2 \cot(\frac{\pi}{2}(\alpha' - \beta'))| \lesssim \int |Z_{t,\alpha'}|^2 d\alpha'. \quad (264)$$

Therefore by the dominated convergence theorem,

$$[Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \Big|_{\partial} = 0. \quad (265)$$

Furthermore we can estimate

$$\left\| [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \quad (266)$$

by Hölder and Hardy's inequality (393), and estimate

$$\left\| \partial_{\alpha'} [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \quad (267)$$

by (408). We may then conclude by (262) and (354) that

$$\mathbb{P}_A [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \Big|_{\partial} = \mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \Big|_{\partial} = 0. \quad (268)$$

Now we return to controlling $\left\| \partial_{\alpha'} \left(\mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right) \right) \right\|_{L^2}$. By (262),

$$\begin{aligned} & \left\| \partial_{\alpha'} \mathbb{P}_A \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right) \right\|_{L^2} = \frac{1}{2} \left\| \partial_{\alpha'} \mathbb{P}_A [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \\ &= \frac{1}{2} \left\| \mathbb{P}_A \partial_{\alpha'} [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \lesssim \left\| \partial_{\alpha'} [Z_t, [Z_t, \mathbb{H}]] \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \\ &\lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2}, \end{aligned} \quad (269)$$

where we have used (339) and (265) to commute the $\partial_{\alpha'}$ inside the \mathbb{P}_A in the second line; then used the L^2 boundedness of \mathbb{P}_A ; and finally used estimate (267) in the last line.

Now we return to controlling part (b) from (254). We begin by decomposing two of the factors into their holomorphic and antiholomorphic projections:

$$\begin{aligned} (b) &= -(I - \mathbb{H}) \left\{ \left(\mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right) \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) D_{\alpha'}^2 \bar{Z}_t \right\} \\ &= -(I - \mathbb{H}) \left\{ \left((-\mathbb{P}_A + \mathbb{P}_H) \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right) \left((\mathbb{P}_A + \mathbb{P}_H) \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right) D_{\alpha'}^2 \bar{Z}_t \right\}. \end{aligned} \quad (270)$$

Observe that $\mathbb{H} = -\mathbb{P}_A + \mathbb{P}_H$. The cross terms cancel, and we're left with

$$(I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right)^2 D_{\alpha'}^2 \bar{Z}_t \right\} - (I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right)^2 D_{\alpha'}^2 \bar{Z}_t \right\}. \quad (271)$$

The second of these terms disappears by (381). We control the first term of (271) by the boundedness of \mathbb{H} and Hölder's inequality, and conclude that

$$\|(b)\|_{L^2} \lesssim \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right\|_{L^\infty}^2, \quad (272)$$

whose second factor we controlled in (146).

We conclude from (254) that

$$\begin{aligned} \|(252)\|_{L^2} &\lesssim (256) + (272) \\ &\leq \|Z_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \left\| \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) \right\|_{L^\infty}^2. \end{aligned} \quad (273)$$

Step 5. Controlling (253). Now we are left with controlling (253), which we expand using $I = \mathbb{P}_A + \mathbb{P}_H$:

$$\begin{aligned} (253) &= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{\bar{Z}_{\alpha'}} (I + \mathbb{H}) D_{\alpha'} Z_t \right\} \\ &= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left((\mathbb{P}_A + \mathbb{P}_H) \frac{Z_t}{\bar{Z}_{\alpha'}} \right) (I + \mathbb{H}) D_{\alpha'} Z_t \right\} \\ &= (I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{\bar{Z}_{\alpha'}} \right) (I + \mathbb{H}) D_{\alpha'} Z_t \right\}, \end{aligned} \quad (274)$$

where the \mathbb{P}_H term disappears by (382). We expand again $I = \mathbb{P}_A + \mathbb{P}_H$ with what remains:

$$(253) = (I - \mathbb{H}) \left\{ ((I + \mathbb{H}) D_{\alpha'} Z_t) (\mathbb{P}_A + \mathbb{P}_H) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{\bar{Z}_{\alpha'}} \right) \right\} \right\}. \quad (275)$$

The \mathbb{P}_H part would be zero by holomorphicity, except that there is a mean term; by (383), what we're left with from that part is, in absolute value,

$$\begin{aligned} \left| \frac{1}{2} \left(\int D_{\alpha'} Z_t \right) \left(\int (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right) \right| &= \left| \frac{1}{2} \left(\int D_{\alpha'} Z_t \right) \left(\int (D_{\alpha'}^2 \bar{Z}_t) \left(\partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right) \right| \\ &\lesssim \|D_{\alpha'} Z_t\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty}. \end{aligned} \quad (276)$$

By (356) and (347) there is no boundary term in the integration by parts.

What's left from (275) to control is the \mathbb{P}_A part,

$$(I - \mathbb{H}) \left\{ ((I + \mathbb{H}) D_{\alpha'} Z_t) \mathbb{P}_A \left((\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right) \right\}. \quad (277)$$

Now we use the boundedness of \mathbb{H} and Hölder to control this in L^2 by

$$\|(I + \mathbb{H}) D_{\alpha'} Z_t\|_{L^\infty} \left\| \mathbb{P}_A \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2}. \quad (278)$$

We rewrite the second factor as a commutator, using (370):

$$\begin{aligned} \left\| \mathbb{P}_A \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right\} \right\|_{L^2} &= \left\| \frac{1}{2} \left[\left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \\ &\lesssim \left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \end{aligned} \quad (279)$$

where we've used the $L^\infty \times L^2$ commutator estimate (398).

We conclude from (275), (276), (278) and (279) that

$$\|(253)\|_{L^2} \lesssim (\|D_{\alpha'} Z_t\|_{L^\infty} + \|(I + \mathbb{H}) D_{\alpha'} Z_t\|_{L^\infty}) \left\| \partial_{\alpha'} (I - \mathbb{H}) \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}. \quad (280)$$

Step 6. *Concluding the estimate for $\|(I - \mathbb{H}) (D_{\alpha'}^2 (\mathcal{A}_t \bar{Z}_{,\alpha'}))\|_{L^2}$ from (227).* We start from (249). We have, from (239), (245), (250), (273) and (280) that

$$\|iii\|_{L^2} \lesssim (245) + (239) + (273) + (280) \lesssim C(E). \quad (281)$$

We must estimate i and ii from (234). When we expand this out carefully, all the problematic terms cancel and what remains can be controlled easily using Hölder and the L^2 boundedness of \mathbb{H} .

We expand i :

$$i = (I - \mathbb{H}) \left(2(D_{\alpha'} Z_t)^2 (D_{\alpha'}^2 \bar{Z}_t) + (2D_{\alpha'}^2 Z_t)(D_{\alpha'} \bar{Z}_{tt}) - 2(D_{\alpha'}^2 Z_t)(D_{\alpha'} Z_t)(D_{\alpha'} \bar{Z}_t) \right). \quad (282)$$

We can therefore estimate

$$\|i\|_{L^2} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \|D_{\alpha'}^2 Z_t\|_{L^2} (\|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty} + \|D_{\alpha'} Z_t\|_{L^\infty}^2). \quad (283)$$

We expand ii :

$$ii = (I - \mathbb{H}) \left(4(D_{\alpha'} Z_t)(D_{\alpha'}^2 \bar{Z}_{tt}) - 8(D_{\alpha'} Z_t)^2 (D_{\alpha'}^2 \bar{Z}_t) - 4(D_{\alpha'} Z_t)(D_{\alpha'} \bar{Z}_t)(D_{\alpha'}^2 Z_t) \right), \quad (284)$$

and we estimate

$$\|ii\|_{L^2} \lesssim \|D_{\alpha'} Z_t\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} + \|D_{\alpha'} Z_t\|_{L^\infty}^2 \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \|D_{\alpha'} Z_t\|_{L^\infty}^2 \|D_{\alpha'}^2 Z_t\|_{L^2}. \quad (285)$$

Combining our various estimates, we have, by (229), that

$$\|(I - \mathbb{H})(D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'}))\|_{L^2} \lesssim (233) + \|i\|_{L^2} + \|ii\|_{L^2} + \|iii\|_{L^2} \lesssim (233) + (283) + (285) + (281) \lesssim C(E). \quad (286)$$

Step 7. Concluding the estimate for G_θ of E_a . Recall that we still needed to control $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}$. Here, we use the Peter-Paul trick Proposition 8. By (223), (224), (226), (227), (286), $\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}$ (49), and $A_1 \geq 1$ (47), we now have an estimate

$$\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \lesssim C(E) + \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}. \quad (287)$$

This implies

$$\begin{aligned} \left\| D_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right) \right\|_{L^2} &\lesssim \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty} \left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} + C(E) + \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \\ &\lesssim C(E) + \|Z_{tt,\alpha'}\|_{L^2} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}, \end{aligned} \quad (288)$$

by (141) and (209). We may now apply Proposition 8. We have

$$\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty} \lesssim (\|Z_{tt,\alpha'}\|_{L^2} + 1) \left(\left\| D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} + C(E) + \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^2} \right) \lesssim C(E), \quad (289)$$

where we have controlled $\|D_{\alpha'} \frac{\mathcal{A}_t}{\mathcal{A}}\|_{L^2}$ in (209) and $\left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ and $\|Z_{tt,\alpha'}\|_{L^2}$ in §5.1.

Combining (214), (215), (223), (224), (226), (227), (286) and (289), we have

$$\|D_{\alpha'}^2(\mathcal{A}_t \bar{Z}_{,\alpha'})\|_{L^2} \lesssim C(E). \quad (290)$$

We conclude by (210), (211), (212), (112) and (290) that

$$\|G_{D_\alpha^2 \bar{z}_t}\|_{L^2(\frac{h_\alpha}{A_1 \circ h})} \leq (211) + (212) \leq (211) + (290) \lesssim C(E). \quad (291)$$

We have now shown that $\frac{d}{dt} E_a$ is bounded by a polynomial of E . This concludes the proof of Theorem 2. \square

10 A characterization of the energy

Our energy is expressed in terms of not only the free surface Z , the velocity Z_t , and their spatial derivatives, but also time derivatives of these quantities. In this section, we give a characterization of our energy in terms of the free surface Z , the velocity Z_t , and their spatial derivatives. We do so in the Riemannian variable, as it directly captures the geometry of the interface Z . We also, in §10.2, offer a heuristic discussion of the singularities inherent in the problem and the crest angles allowed by our energy, as indicated by the Riemann mapping.

10.1 A characterization of the energy in terms of position and velocity

In this section, we translate the terms of our energy involving time derivatives into terms depending only on the free surface Z , the velocity Z_t , and their spatial derivatives. We do this using the basic identity (49)

$$\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}, \quad (292)$$

(45), and the holomorphicity of \bar{Z}_t and various other quantities discussed in §A.3. These basic water wave equations allow us to show that quantities involving \bar{Z}_{tt} can be controlled by analogous quantities involving $\frac{1}{Z_{,\alpha'}}$, along with various lower-order terms.²⁷

²⁷We remark that for these estimates we *do not* ever rely on (high order) $\dot{H}^{1/2}$ parts of the energies.

The estimate we prove is

$$E(t) \leq C \left(\|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}, \right. \\ \left. \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}, \left\| \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{1/2}}, \left\| \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \right), \quad (293)$$

where the constant depends polynomially on its terms. We remark that this inequality can be reversed: each of the factors on the RHS of (293) is controlled by the energy. That is,

$$\|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}, \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}, \left\| \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \\ \|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{1/2}}, \left\| \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \lesssim C(E(t)). \quad (294)$$

Therefore, these quantities fully characterize our energy. In the proof of our a priori estimate, we have shown (294) for every term except $\left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}$, which we never had a need to control. One can adapt the argument in §10.1.3 below to show that $\left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}$ can be controlled by the energy.²⁸

We remark because both \bar{Z}_t and $\frac{1}{\bar{Z}_{,\alpha'}}$ are the boundary values of periodic holomorphic functions, the weighted derivative $D_{\alpha'}$ corresponds to the complex derivative ∂_z , or the gradient of the corresponding quantities in the spatial domain P^- . We also note that $\bar{Z}_{,\alpha'} = (\Phi^{-1})_z$ is a natural geometric quantity well-suited to this problem: it captures the geometry of the free surface directly through the Riemann mapping $\Phi^{-1} : P^- \rightarrow \Omega(t)$.

10.1.1 The proof

Throughout the following proof we will rely on the fact that $A_1 \geq 1$ (47), the estimate (93)

$$\|A_1\|_{L^\infty} \lesssim 1 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2, \quad (296)$$

the Sobolev estimate (94)

$$\|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \lesssim \|\bar{Z}_{t,\alpha'}\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \quad (297)$$

and the estimate

$$\left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty} \lesssim \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}, \quad (298)$$

which holds by Sobolev inequality (386).²⁹

²⁸To do this, it comes down once again to estimating $\left\| \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2}$, except this time we need to do this without the dependence on $\left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}$. That dependence comes from estimate (318). (It also comes from using $\left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}$ in the Sobolev inequality for $\left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty}$; this is not a problem, since $\left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty}$ is controlled by the energy.) To handle (318), we take advantage of the fact that $(I - \mathbb{H}) \left\{ \partial_{\alpha'} D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\} = 0$ (this is due to (362), (368) and the second principle in §A.3) to rewrite the problematic term as a commutator and then use commutator estimate (394):

$$\left\| (I - \mathbb{H}) \left\{ \frac{A_1}{\bar{Z}_{,\alpha'}} \partial_{\alpha'} D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\} \right\|_{L^2} = \left\| [\bar{Z}_{tt}, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \lesssim \|\bar{Z}_{tt,\alpha'}\|_{L^2} \left\| D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^\infty}, \quad (295)$$

both of which are controlled by the energy.

²⁹Note that $f \left(D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right)^2 = 0$ by the same argument that was used at (384) to show $f(D_{\alpha'} \bar{Z}_t)^2 = 0$.

We begin by noting that it suffices to control only the first terms of E_a and E_b , since the remaining terms of the energy are (up to a factor of A_1) already on the RHS of (293).

For the first term of E_a , by the commutator identity (414),

$$\begin{aligned} \int |\partial_t D_\alpha^2 \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha &\lesssim \int |D_\alpha^2 \bar{z}_{tt}|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha + \int |[\partial_t, D_\alpha^2] \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\ &\lesssim \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}^2 + \int |(D_\alpha z_t) D_\alpha^2 \bar{z}_t + (D_\alpha^2 z_t) D_\alpha \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \\ &\lesssim \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}^2 + \|D_\alpha z_t\|_{L^\infty}^2 (\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}^2 + \|D_{\alpha'}^2 Z_t\|_{L^2}^2). \end{aligned} \quad (299)$$

By (100) and (391),

$$\|D_{\alpha'}^2 Z_t\|_{L^2} \lesssim \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}. \quad (300)$$

We conclude that

$$\int |\partial_t D_\alpha^2 \bar{z}_t|^2 \frac{h_\alpha}{A_1 \circ h} d\alpha \lesssim C \left(\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} \right). \quad (301)$$

For the first term of E_b , we use the commutator identity (413) to get

$$\begin{aligned} \int |\partial_t D_\alpha \bar{z}_t|^2 \frac{1}{\mathfrak{a}} d\alpha &\lesssim \int |D_\alpha \bar{z}_{tt}|^2 \frac{1}{\mathfrak{a}} d\alpha + \int |[\partial_t, D_\alpha] \bar{z}_t|^2 \frac{1}{\mathfrak{a}} d\alpha \\ &\lesssim \int |D_\alpha \bar{z}_{tt}|^2 \frac{(A_1 \circ h)}{\mathfrak{a}} d\alpha + \int |(D_\alpha z_t) D_\alpha \bar{z}_t|^2 \frac{(A_1 \circ h)}{\mathfrak{a}} d\alpha \\ &\lesssim \|\bar{Z}_{tt,\alpha'}\|_{L^2}^2 + \|D_\alpha z_t\|_{L^\infty}^2 \|\bar{Z}_{t,\alpha'}\|_{L^2}^2 \leq C (\|\bar{Z}_{tt,\alpha'}\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}). \end{aligned} \quad (302)$$

All that remains to do from (301) and (302) is to estimate $\|\bar{Z}_{tt,\alpha'}\|_{L^2}$ and $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$ in terms of Z_t and $\frac{1}{\bar{Z}_{,\alpha'}}$, which we now do, in §10.1.2 and §10.1.3.

10.1.2 Controlling $\|\bar{Z}_{tt,\alpha'}\|_{L^2}$

Using (292), we estimate

$$\|\partial_{\alpha'} \bar{Z}_{tt}\|_{L^2} \lesssim \|A_1\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} + \|D_{\alpha'} A_1\|_{L^2}. \quad (303)$$

To control $\|D_{\alpha'} A_1\|_{L^2}$, we follow a similar procedure to what we did in (116)-(117), except instead of using $\bar{Z}_{tt} - i$, we use $\frac{1}{\bar{Z}_{,\alpha'}}$ and estimate things in terms of $\frac{1}{\bar{Z}_{,\alpha'}}$. We get

$$\|D_{\alpha'} A_1\|_{L^2} \lesssim \|\bar{Z}_{t,\alpha'}\|_{L^2}^2 \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}. \quad (304)$$

Combining (303) and (304) we conclude that

$$\|\partial_{\alpha'} \bar{Z}_{tt}\|_{L^2} \leq C \left(\|\bar{Z}_{t,\alpha'}\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}} \right\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \right). \quad (305)$$

10.1.3 Controlling $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$

From (292), we have

$$iD_{\alpha'}^2 \bar{Z}_{tt} = \underbrace{A_1 D_{\alpha'}^2 \frac{1}{\bar{Z}_{,\alpha'}} + 2(D_{\alpha'} A_1) D_{\alpha'} \frac{1}{\bar{Z}_{,\alpha'}}}_{e_1} + \frac{1}{\bar{Z}_{,\alpha'}} D_{\alpha'}^2 A_1. \quad (306)$$

We estimate $\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}$ through the following procedure. First we note that the only challenging term to control on the RHS of (306) is the last one, $\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1$. We observe that this is almost real, modulo factors of $\frac{1}{\bar{Z}_{\alpha'}}$ and its derivatives. Therefore, we will be able to use the $\Re(I - \mathbb{H})$ trick and, through a series of commutators, reduce the estimate for $\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1$ to an estimate of $(I - \mathbb{H})(D_{\alpha'}^2 \frac{A_1}{\bar{Z}_{\alpha'}}) = (I - \mathbb{H})(iD_{\alpha'}^2 \bar{Z}_{tt})$. Since \bar{Z}_t is holomorphic, we will be able to rewrite $(I - \mathbb{H})(iD_{\alpha'}^2 \bar{Z}_{tt})$ in terms of commutators and obtain favorable estimates. We now give the details.

We first estimate the error term e_1 in (306):

$$\begin{aligned} \|e_1\|_{L^2} &\lesssim \|A_1\|_{L^\infty} \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} + \|D_{\alpha'} A_1\|_{L^2} \left\| D_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^\infty} \\ &\lesssim (1 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2) \left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} + (304) \left(\left\| D_{\alpha'}^2 \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \right). \end{aligned} \quad (307)$$

It remains to control $\left\| \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2}$. We want to use $(I - \mathbb{H})$ to turn our quantity into commutators, but to do so we need to factor $D_{\alpha'}$ into a real-weighted derivative $|D_{\alpha'}| := \frac{1}{|\bar{Z}_{\alpha'}|} \partial_{\alpha'}$ so that we may invert $(I - \mathbb{H})$. From (216), we have

$$\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 = \frac{1}{\bar{Z}_{\alpha'}} \left(\frac{|Z_{\alpha'}|}{\bar{Z}_{\alpha'}} \right)^2 |D_{\alpha'}|^2 A_1 + \underbrace{\frac{1}{\bar{Z}_{\alpha'}} \frac{|Z_{\alpha'}|}{\bar{Z}_{\alpha'}} \left(|D_{\alpha'}| \frac{|Z_{\alpha'}|}{\bar{Z}_{\alpha'}} \right) |D_{\alpha'}| A_1}_{e_2}. \quad (308)$$

We multiply both sides by $\left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3$ so that the first term on the RHS is purely real:

$$\left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 = \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} \left(\frac{|Z_{\alpha'}|}{\bar{Z}_{\alpha'}} \right)^2 |D_{\alpha'}|^2 A_1 + \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 e_2. \quad (309)$$

Now we apply $\Re(I - \mathbb{H})$ to each side, and conclude from the fact that $A_1 \in \mathbb{R}$ that

$$\left| \frac{1}{\bar{Z}_{\alpha'}} |D_{\alpha'}|^2 A_1 \right| \lesssim \left| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 \right\} \right| + \left| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 e_2 \right\} \right|. \quad (310)$$

We conclude from (308) and (310) that

$$\left\| \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2} \lesssim \|e_2\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2}. \quad (311)$$

By (391) and (304) we estimate

$$\|e_2\|_{L^2} \lesssim \left\| D_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \lesssim (298)(304). \quad (312)$$

It remains to estimate $\left\| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2}$. To get the right commutator estimate, we first rewrite this as

$$\begin{aligned} \left\| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'}^2 A_1 \right\} \right\|_{L^2} &\lesssim \left\| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} \partial_{\alpha'} \left(\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} \\ &\quad + \left\| (I - \mathbb{H}) \left\{ \left(\frac{|Z_{\alpha'}|}{|\bar{Z}_{\alpha'}|} \right)^3 \frac{1}{\bar{Z}_{\alpha'}} \left(\partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right) D_{\alpha'} A_1 \right\} \right\|_{L^2}. \end{aligned} \quad (313)$$

We estimate the second term on the RHS of (313) directly:

$$\begin{aligned} \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{\alpha'}}{|Z_{\alpha'}|} \right)^3 \frac{1}{Z_{\alpha'}} \left(\partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) D_{\alpha'} A_1 \right\} \right\|_{L^2} &\lesssim \left\| D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \\ &\lesssim (298)(304). \end{aligned} \quad (314)$$

For the first term on the RHS of (313), we commute the factor $\left(\frac{Z_{\alpha'}}{|Z_{\alpha'}|} \right)^3$ outside the $(I - \mathbb{H})$, bringing along $\frac{1}{Z_{\alpha'}}$ to ensure that the commutator is controllable, and then bringing the $\frac{1}{Z_{\alpha'}}$ back inside:

$$\begin{aligned} \left\| (I - \mathbb{H}) \left\{ \left(\frac{Z_{\alpha'}}{|Z_{\alpha'}|} \right)^3 \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} &\lesssim \left\| \left[\left(\frac{Z_{\alpha'}}{|Z_{\alpha'}|} \right)^3 \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \\ &+ \left\| \left[\frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2}. \end{aligned} \quad (315)$$

We estimate the first two terms on the RHS of (315) using commutator estimate (394)³⁰:

$$\begin{aligned} \left\| \left[\left(\frac{Z_{\alpha'}}{|Z_{\alpha'}|} \right)^3 \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} &+ \left\| \left[\frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\|_{L^2} \\ &\lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^2} \left\| \frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}. \end{aligned} \quad (316)$$

We will postpone estimating $\left\| \frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}$ until the end of this long series of calculations. For the moment, we take the last term from the RHS of (315):

$$\begin{aligned} \left\| (I - \mathbb{H}) \left\{ \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \left(\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1 \right) \right\} \right\|_{L^2} \\ \lesssim \left\| (I - \mathbb{H}) D_{\alpha'}^2 \left(\frac{1}{Z_{\alpha'}} A_1 \right) \right\|_{L^2} + \left\| (I - \mathbb{H}) \left\{ D_{\alpha'} \left(A_1 D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\} \right\|_{L^2}. \end{aligned} \quad (317)$$

We estimate the second term by

$$\begin{aligned} \left\| (I - \mathbb{H}) \left\{ D_{\alpha'} \left(A_1 D_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\} \right\|_{L^2} &\lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_{L^2} \|A_1\|_{L^\infty} + \left\| D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{L^\infty} \|D_{\alpha'} A_1\|_{L^2} \\ &\lesssim \left\| D_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_{L^2} (1 + \|\bar{Z}_{t,\alpha'}\|_{L^2}^2) + (298)(304). \end{aligned} \quad (318)$$

Finally, for the first term on the RHS of (317), we recall from (42) that

$$\frac{1}{Z_{\alpha'}} A_1 = i(\bar{Z}_{tt} - i) = i(Z_t D_{\alpha'} \bar{Z}_t) + iF_t \circ Z + 1 \quad (319)$$

for $F(z(\alpha, t), t) := \bar{z}_t(\alpha, t)$. We apply $(I - \mathbb{H}) D_{\alpha'}^2$ to this, and the last two terms disappear. We get

$$\begin{aligned} \left\| (I - \mathbb{H}) D_{\alpha'}^2 \left(\frac{1}{Z_{\alpha'}} A_1 \right) \right\|_{L^2} &= \left\| (I - \mathbb{H}) D_{\alpha'}^2 (Z_t D_{\alpha'} \bar{Z}_t) \right\|_{L^2} \\ &\lesssim \left\| (I - \mathbb{H}) \{ (D_{\alpha'}^2 Z_t) D_{\alpha'} \bar{Z}_t \} \right\|_{L^2} + \left\| (I - \mathbb{H}) \{ (D_{\alpha'} Z_t) (D_{\alpha'}^2 \bar{Z}_t) \} \right\|_{L^2} \\ &+ \left\| (I - \mathbb{H}) \left\{ \frac{Z_t}{Z_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \right\|_{L^2}. \end{aligned} \quad (320)$$

³⁰Note that this does not require that $\frac{1}{Z_{\alpha'}} D_{\alpha'} A_1$ has periodic boundary behavior; this explains why we chose this commutator and moved $\frac{1}{Z_{\alpha'}}$ around instead of using the $L^\infty \times L^2$ estimate. Cf. footnote 26.

We estimate the first two terms directly by

$$\begin{aligned} & \|(I - \mathbb{H}) \{ (D_{\alpha'}^2 Z_t) D_{\alpha'} \bar{Z}_t \} \|_{L^2} + \|(I - \mathbb{H}) \{ (D_{\alpha'} Z_t) (D_{\alpha'}^2 \bar{Z}_t) \} \|_{L^2} \\ & \lesssim \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} (\|D_{\alpha'}^2 Z_t\|_{L^2} + \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}) \lesssim (297)(\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + (300)). \end{aligned} \quad (321)$$

We are left with the last term on the RHS of (320). We first decompose $\frac{Z_t}{\bar{Z}_{\alpha'}}$ into its holomorphic and antiholomorphic projections. The term with the holomorphic projection disappears by (378). With what remains, we use (370) to get a commutator, which we control by commutator estimate (398):

$$\begin{aligned} & \left\| (I - \mathbb{H}) \left\{ \frac{Z_t}{\bar{Z}_{\alpha'}} \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \right\|_{L^2} = \left\| (I - \mathbb{H}) \left\{ \left(\mathbb{P}_A \frac{Z_t}{\bar{Z}_{\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\} \right\|_{L^2} \\ & = \left\| \left[\left(\mathbb{P}_A \frac{Z_t}{\bar{Z}_{\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \right\|_{L^2} \lesssim \left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{\bar{Z}_{\alpha'}} \right\|_{L^\infty} \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \\ & \lesssim \left(\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \|Z_{t,\alpha'}\|_{L^2} \left(1 + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \right) \right) \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} \end{aligned} \quad (322)$$

by (147) and (297).

We now give the estimate for $\left\| \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty} = \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_{L^\infty}$ in (316). We do so using (116). We have

$$\begin{aligned} \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 &= \Im \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \left(\frac{1}{|Z_{\alpha'}|^2} - \frac{1}{|Z_{\beta'}|^2} \right) \bar{Z}_{t,\beta'}(\beta') d\beta \\ &+ \Im \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \frac{1}{\bar{Z}_{\beta'}} D_{\beta'} \bar{Z}_t(\beta') d\beta \\ &= I + II. \end{aligned} \quad (323)$$

From Hölder's inequality, Hardy's inequality (393) and the mean value theorem,³¹ we have

$$\|II\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \left\| D_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^\infty} \lesssim \|Z_{t,\alpha'}\|_{L^2}^2 \quad (298). \quad (324)$$

We rewrite II using integration-by-parts identity (395):

$$\begin{aligned} & \frac{1}{2i} \int \frac{\pi (Z_t(\alpha') - Z_t(\beta'))}{2 \sin^2(\frac{\pi}{2}(\alpha' - \beta'))} \frac{1}{\bar{Z}_{\beta'}} D_{\beta'} \bar{Z}_t(\beta') d\beta \\ &= -[Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'} \bar{Z}_t \right) + \mathbb{H} \left(Z_{t,\alpha'} \frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'} \bar{Z}_t \right) \\ &= -[Z_t, \mathbb{H}] \partial_{\alpha'} \left(\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'} \bar{Z}_t \right) - \left[\frac{1}{\bar{Z}_{\alpha'}} \overline{D_{\alpha'} \bar{Z}_t}, \mathbb{H} \right] \bar{Z}_{t,\alpha'} + |D_{\alpha'} \bar{Z}_t|^2. \end{aligned} \quad (325)$$

Using (407) on the first two terms on the RHS of (325), we get

$$\begin{aligned} \|II\|_{L^\infty} &\lesssim \|Z_{t,\alpha'}\|_{L^2} \left\| \partial_{\alpha'} \left(\frac{1}{\bar{Z}_{\alpha'}} D_{\alpha'} \bar{Z}_t \right) \right\|_{L^2} + \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}^2 \\ &\lesssim \|Z_{t,\alpha'}\|_{L^2} \left(\|D_{\alpha'}^2 \bar{Z}_t\|_{L^2} + \left\| \partial_{\alpha'} \frac{1}{\bar{Z}_{\alpha'}} \right\|_{L^2} \|D_{\alpha'} \bar{Z}_t\|_{L^\infty} \right) + \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}^2. \end{aligned} \quad (326)$$

³¹Note that $\frac{1}{|Z_{\alpha'}|^2}$ is periodic.

Combining (323), (324), (326), (298), and (297), we have

$$\begin{aligned} \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_{L^\infty} &\leq \|I\|_{L^\infty} + \|II\|_{L^\infty} \\ &\leq C \left(\|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \right). \end{aligned} \quad (327)$$

We now sum up these estimates. From (311), and using (327) to estimate $\left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'} A_1 \right\|_{L^\infty}$, we have

$$\begin{aligned} \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 A_1 \right\|_{L^2} &\lesssim (312) + (314) + (316) + (318) + (321) + (322) \\ &\lesssim C \left(\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \right). \end{aligned} \quad (328)$$

Combining (307) and (328) we conclude that

$$\|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2} \lesssim C \left(\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \right). \quad (329)$$

10.2 Singularities and the angle of the crest

In §10.1 at (293) and (294) we characterized our energy in terms of various L^2 and $\dot{H}^{1/2}$ norms of quantities in Riemannian coordinates (as well as a single quantity, $\frac{1}{Z_{,\alpha'}}$, in L^∞). When there is a non-right angle ν at the corner, or when there is a singularity in the middle of the free surface, the Riemann mapping will have a singularity. In this section, we discuss this singularity and what it suggests about the angle ν or the interior angle of an angled crest in the middle of the free surface, continuing the discussion from §3.2.2. We emphasize that unlike the rest of the paper, which is fully rigorous, the discussion in this section is heuristic and is under various unstated assumptions.

As in §3.2.2, we will phrase our discussion in terms of a singularity at the corner, but it applies more broadly to singularities in the middle of the free surface.³² We thus henceforth focus on the angle ν at the corner. Of course, our energy is finite in the regime when $\nu = \frac{\pi}{2}$, so we can focus on the case where $\nu < \frac{\pi}{2}$.³³

Throughout this section, we will abuse notation and say, e.g., $\Phi(z) \approx z^r$ or $h(\alpha) \approx \alpha^r$ at the corner, when in fact the corners are at $z, \alpha = \pm 1$, not 0.

If ν is the angle the water wave makes with the wall, the Riemann mapping $\Phi(z)$ should behave like z^r at the corner, where $r\nu = \frac{\pi}{2}$. For $\nu < \frac{\pi}{2}$, we have $r > 1$.

Recall from (293) and (294) that among the quantities that characterize the energy, $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$, $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$ and $\left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ are the terms purely related to the surface. We will see through the following calculation what non-right angles ν are allowed if $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$, $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$ and $\left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ are finite.

Note that $Z(\alpha') = \Phi^{-1}(\alpha') \approx (\alpha')^{1/r}$ so $Z_{,\alpha'} = \partial_{\alpha'}(\Phi^{-1}) \approx (\alpha')^{1/r-1}$, and

$$\frac{1}{Z_{,\alpha'}} \approx (\alpha')^{1-1/r}, \quad \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \approx (\alpha')^{-1/r} \quad (r \neq 1). \quad (330)$$

Therefore, assuming $r > 1$, $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \in L^2$ if and only if $r > 2$ if and only if $\nu < \frac{\pi}{4}$. Similarly, $D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \in L^2$ so long as $r > 2$.³⁴

³²The angled crests in the middle of the free surface don't have to be symmetric.

³³Recall from the discussion in §3.2.2 that we cannot have $\nu > \frac{\pi}{2}$ in our energy regime.

³⁴We remark that, even though E_a (which roughly includes $\left\| D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$) is higher-order than E_b (which roughly includes $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$) in terms of the number of derivatives, the two energies are comparable in the sense that they allow precisely the same angles.

We conclude from this discussion that our energy will be finite only when $\nu < \frac{\pi}{4}$ (or when $\nu = \frac{\pi}{2}$). This coincides precisely with the angles in the self-similar solutions of [Wu12].³⁵ For singularities in the middle of the free surface, this suggests that the interior angle must be less than $\frac{\pi}{2}$.³⁶

In the following discussion, we work under the assumption that $0 < c_1 \leq |z_\alpha| \leq c_2 < \infty$ for all time in our timeframe, so the function $z(\alpha)$ doesn't affect the singularity (to first order).³⁷

Since $z \approx \alpha$, $|z_\alpha| \approx 1$, we have

$$h(\alpha) = \Phi(z(\alpha)) \approx \alpha^r, \quad h^{-1}(\alpha') \approx (\alpha')^{1/r}. \quad (331)$$

On differentiating (331), we get that

$$h_\alpha(\alpha) \approx \alpha^{r-1}, \quad h_\alpha \circ h^{-1}(\alpha') \approx (\alpha')^{\frac{1}{r}(r-1)} = (\alpha')^{1-\frac{1}{r}}. \quad (332)$$

The behavior of the angle ν over time is of significant interest. This angle is determined by z_α at the corner. Therefore, the behavior of $z_{t\alpha}$ and $z_{tt\alpha}$ at the corner should determine how the angle changes. Since $Z_{t,\alpha'} = \frac{z_{t\alpha}}{h_\alpha} \circ h^{-1}$ and $\frac{1}{h_\alpha} \circ h^{-1} \approx (\alpha')^{1/r-1}$, we must have $z_{t\alpha} \rightarrow 0$ at the corner if $Z_{t,\alpha'} \in L^2$, as our energy assumes. A similar argument holds for $z_{tt\alpha}$. This suggests that if initially $\nu < \frac{\pi}{4}$, the angle would not change while the energy remained finite. This holds true also for interior angles at the angled crests. Careful analysis of such behavior at the corner would be an avenue for future research.

Appendices

A Holomorphicity, mean and boundary behavior

A.1 The Hilbert transform \mathbb{H}

Recall that in §3 we introduced the Hilbert transform \mathbb{H} :

$$\mathbb{H}f(\alpha') := \frac{1}{2i} \text{pv} \int_I \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta') d\beta', \quad \text{for } \alpha' \in [-1, 1]. \quad (333)$$

We know from Proposition 1 that a function g is the boundary value of a periodic holomorphic function on P^- if and only if $(I - \mathbb{H})g = \oint_I g$, and that for any function f , $(I + \mathbb{H})f$ is the boundary value of a periodic holomorphic function on P^- , with $(I - \mathbb{H})(I + \mathbb{H})f = \oint_I f$. Recall also that we defined at (38) the holomorphic and antiholomorphic projection operators

$$\mathbb{P}_A f := \frac{(I - \mathbb{H})}{2} f; \quad \mathbb{P}_H f := \frac{(I + \mathbb{H})}{2} f. \quad (334)$$

Here we gather some basic properties of the Hilbert transform \mathbb{H} that will be used in this paper:

Proposition 3. *a. Let $1 < p < \infty$. Then there exists $C_p < \infty$ such that for all $f \in L^p$*

$$\|\mathbb{H}f\|_{L^p} \leq C_p \|f\|_{L^p}. \quad (335)$$

b. Let $f \in L^p$ for some $p > 1$. Then

$$\mathbb{H}^2 f = f - \oint_I f; \quad \mathbb{P}_A \mathbb{P}_H f = \mathbb{P}_H \mathbb{P}_A f = \frac{1}{4} \oint_I f. \quad (336)$$

³⁵We recall that our energy is finite for these solutions.

³⁶We note that our energy does not apply to Stokes waves of maximum height (interior angle = $\frac{2\pi}{3}$).

³⁷This holds so long as our energy is finite; see footnote 10.

c. Let $f \in L^p$, $g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Then

$$\int f(\mathbb{H}g) = - \int (\mathbb{H}f)g; \quad \int (\mathbb{P}_A f)g = \int f(\mathbb{P}_H g). \quad (337)$$

d. Let $f \in L^p$, $g \in L^q$ for $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} < 1$. Then

$$\mathbb{P}_A \{(\mathbb{P}_H f)(\mathbb{P}_H g)\} = \frac{1}{8} \left(\oint f \right) \left(\oint g \right). \quad (338)$$

Parts a., b., and c. are classical results. We know the product of periodic holomorphic functions is periodic holomorphic. Part d. is an easy consequence of Proposition 1, parts b. and c., and the fact that $(\mathbb{P}_H f)(\mathbb{P}_H g)$ is the boundary value of a periodic holomorphic function on P^- .

We also note the following commutator:

Proposition 4. Let $f \in C^0[-1, 1]$ with $f' \in L^p$ for some $p > 1$. If $f|_{\partial} = 0$, then

$$[\partial_{\alpha'}, \mathbb{H}]f = 0. \quad (339)$$

This proposition can be verified easily by integration by parts.

Finally, we will need the following proposition.

Proposition 5. Let $f \in L^\infty$, $g \in L^p$ for some $p > 1$. Suppose $(I - \mathbb{H})f = \oint f$. Then

$$[f, \mathbb{H}]g = \frac{1}{2}(I + \mathbb{H})([f, \mathbb{H}]g) - \frac{1}{2}\oint(fg) + \frac{1}{2}\left(\oint f\right)\left(\oint g\right). \quad (340)$$

Proof. We begin by observing that

$$[f, \mathbb{H}]g = f(I + \mathbb{H})g - (I + \mathbb{H})(fg). \quad (341)$$

Because $(I - \mathbb{H})f = \oint f$, by Proposition 1, $[f, \mathbb{H}]g$ is the boundary value of a periodic holomorphic function on P^- . Therefore,

$$\begin{aligned} (I - \mathbb{H})([f, \mathbb{H}]g) &= \oint [f, \mathbb{H}]g = \oint f(I + \mathbb{H})g - \oint fg \\ &= \oint ((I - \mathbb{H})f)g - \oint (fg) = \left(\oint f\right)\left(\oint g\right) - \oint(fg) \end{aligned} \quad (342)$$

by (337). □

A.2 Boundary properties

Because we are working on a compact domain, we have to worry about boundary terms when integrating by parts. Furthermore, various results involving the periodic Hilbert transform require that certain functions be periodic. Therefore, it is critical that various potentially problematic boundary terms either disappear or can be controlled. For this reason, we need to take care in specifying what the boundary properties of our various functions are.

Our hope is that we are dealing with functions $f(\alpha, t)$ that satisfy

$$f|_{\partial} := f(1, t) - f(-1, t) = 0. \quad (343)$$

Recall that in §2.3 we have shown that our basic function \bar{z}_t satisfies $\bar{z}_t|_{\partial} = 0$ (22). Because this boundary difference is constant over time, and doesn't change under conjugation,

$$\bar{z}_t|_{\partial} = 0, \quad \bar{z}_{tt}|_{\partial} = 0, \quad z_t|_{\partial} = 0, \quad z_{tt}|_{\partial} = 0. \quad (344)$$

Recall also that D_α preserves holomorphicity and periodic boundary behavior of holomorphic functions, so $D_\alpha^k \bar{z}_t|_\partial = 0$ for any $k \geq 0$ (24). For the same reason, we have

$$\partial_t D_\alpha^k \bar{z}_t|_\partial = 0, \quad \overline{D_\alpha^k \bar{z}_t}|_\partial = 0, \quad \overline{\partial_t D_\alpha^k \bar{z}_t}|_\partial = 0, \quad k \geq 0. \quad (345)$$

Now by applying change of coordinates to (344) and (345), we have

$$\bar{Z}_t|_\partial = \bar{Z}_{tt}|_\partial = Z_t|_\partial = Z_{tt}|_\partial = 0 \quad (346)$$

and

$$D_{\alpha'}^k \bar{Z}_t|_\partial = \overline{D_{\alpha'}^k \bar{Z}_t}|_\partial = 0 \text{ for } k \geq 0. \quad (347)$$

We caution that this periodic boundary behavior does *not* necessarily hold for $D_{\alpha'}^k \bar{Z}_{tt}$ or $D_{\alpha'}^k Z_t$, because the holomorphicity is crucial and disappears under differentiation by t or conjugation.

We do, however, have

$$\Re D_{\alpha'} Z_t|_\partial = 0. \quad (348)$$

This is used in §7. Here, the \Re is the savior. Note that $\Re D_{\alpha'} \bar{Z}_t$ is even and $\Im D_{\alpha'} \bar{Z}_t$ is odd; this is because $D_{\alpha'}$ flips the parity of the real and imaginary parts, and $\Re \bar{Z}_t$ is odd and $\Im \bar{Z}_t$ is even. On the other hand, we know $\Re \frac{\bar{Z}_{,\alpha'}}{Z_{,\alpha'}}$ is even and $\Im \frac{\bar{Z}_{,\alpha'}}{Z_{,\alpha'}}$ is odd. We write

$$\Re(D_{\alpha'} Z_t) = \Re\left(\frac{\bar{Z}_{,\alpha'}}{Z_{,\alpha'}} \overline{D_{\alpha'} \bar{Z}_t}\right) = \left(\Re \frac{\bar{Z}_{,\alpha'}}{Z_{,\alpha'}}\right) (\Re D_{\alpha'} \bar{Z}_t) + \left(\Im \frac{\bar{Z}_{,\alpha'}}{Z_{,\alpha'}}\right) (\Im D_{\alpha'} \bar{Z}_t). \quad (349)$$

Each of the terms on the RHS of (349) are even. This concludes the proof of (348).

We list a few other boundary properties that we need in the following. We have:

$$\frac{1}{Z_{,\alpha'}} \Big|_\partial = 0; \quad (350)$$

this follows from (32) and the symmetry of the Riemann Mapping Φ .

We also can state the following:

$$\frac{1}{Z_{,\alpha'}}(\pm 1) \equiv 0 \text{ or } \left\{ \text{the angle } \nu \text{ is } 90^\circ \text{ and } \frac{z_\alpha}{|z_\alpha|} \Big|_\partial = 0 \right\}. \quad (351)$$

This holds because $\frac{1}{Z_{,\alpha'}} = 0$ when $\nu < \frac{\pi}{2}$; otherwise, $\nu = \frac{\pi}{2}$ and so $y_\alpha(\pm 1) \equiv 0$, and thus $z_\alpha|_\partial = 0$.

We at one point need the fact that

$$(\bar{Z}_{ttt} + i \mathcal{A} \bar{Z}_{t,\alpha'})|_\partial = \bar{Z}_{ttt} + (Z_{tt} + i) D_{\alpha'} \bar{Z}_t|_\partial = 0. \quad (352)$$

This follows from the fact that each of the factors \bar{Z}_{ttt} , $(Z_{tt} + i)$, and $D_{\alpha'} \bar{Z}_t$ satisfies the periodic boundary conditions.

We also at one point need the fact that

$$\frac{h_{t\alpha}}{h_\alpha} \circ h^{-1} \Big|_\partial = 0. \quad (353)$$

This follows because h and therefore h_t are odd, and ∂_α flips parity.

We know periodic boundary behavior is preserved by the Hilbert transform \mathbb{H} :

Proposition 6. *Let $f \in C^0[-1, 1]$, with $f' \in L^2[-1, 1]$, and $f|_\partial = 0$. Then*

$$(\mathbb{H}f)|_\partial = 0. \quad (354)$$

We note that functions f satisfying $f|_{\partial} = 0$ form an algebra; any product of them retain this behavior. In particular, we often use

$$\frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial} = 0, \quad (355)$$

which follows from (346) and (350). From Proposition 6, we can conclude that

$$\mathbb{H} \frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial}, \quad \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial}, \quad \mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \Big|_{\partial} = 0. \quad (356)$$

To apply the proposition, we rely on the fact that $\partial_{\alpha'} \frac{Z_t}{Z_{,\alpha'}} \in L^2$, which is true in our energy regime since $D_{\alpha} Z_t \in L^{\infty}$, $Z_{t,\alpha'} \in L^2$, $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \in L^2$, $\frac{1}{Z_{,\alpha}} \in L^{\infty}$,³⁸ along with a priori assumption $Z_t \in C^0$ and the boundary condition (355).

Finally, let $F = \bar{v}$ be the conjugate of the velocity, as defined in §3.2.1. We know that F is periodic holomorphic on $\Omega(t)$, therefore so are $F_t, F_{tz}, F_{tt}, F_{tzz}$, etc. We show in §3.2.3 that

$$F_t \circ z = \bar{z}_{tt} - (D_{\alpha} \bar{z}_t) z_t, \quad F_{tz} \circ z = D_{\alpha} (\bar{z}_{tt} - (D_{\alpha} \bar{z}_t) z_t). \quad (357)$$

Therefore

$$F_t \circ Z|_{\partial} = 0; \quad F_{tz} \circ z|_{\partial} = D_{\alpha} ((\bar{z}_{tt} - (D_{\alpha} \bar{z}_t) z_t))|_{\partial} = 0; \quad F_{ttz} \circ Z|_{\partial} = 0; \quad (358)$$

$$D_{\alpha'} (F_{tz} \circ Z)|_{\partial} = F_{tzz} \circ Z|_{\partial} = 0. \quad (359)$$

A.3 Holomorphic functions and what disappears under $(I - \mathbb{H})$

In this section, we note which of the functions we are dealing with are the boundary values of periodic holomorphic functions—and which, further, have mean zero and thus disappear under $(I - \mathbb{H})$. From Proposition 1 we know that, to show that $(I - \mathbb{H})$ of various functions disappears, it suffices to show that they are boundary values of periodic holomorphic functions and to show that their means are zero.

We rely fundamentally on the following facts:

First, we have that the conjugate velocity is holomorphic, and goes to zero as $y \rightarrow -\infty$ by (7), so

$$(I - \mathbb{H}) \bar{Z}_t = 0. \quad (360)$$

Then we have three identities about the Riemann mapping. Recall that

$$Z_{,\alpha'} = \partial_{\alpha'} \Phi^{-1}(\alpha', t); \quad \frac{1}{Z_{,\alpha'}} = \Phi_z \circ Z. \quad (361)$$

Both of these are clearly holomorphic. Therefore, we have

$$(I - \mathbb{H}) \frac{1}{Z_{,\alpha'}} = \oint \frac{1}{Z_{,\alpha'}}, \quad (I - \mathbb{H}) Z_{,\alpha'} = 1. \quad (362)$$

The mean $\oint \frac{1}{Z_{,\alpha'}} = 1$ by the fundamental theorem of calculus, since $Z(1, t) = 1, Z(-1, t) = -1$ for all time.

Finally, we have

$$(I - \mathbb{H}) \{\Phi_t \circ Z\} = \oint_I \Phi_t \circ Z. \quad (363)$$

Here, we have $\Phi_t \circ \Phi^{-1}$ is holomorphic because it is the limit of holomorphic functions, and we know that Φ_t is periodic by the Schwarz reflection.

From these facts, we will be able to deduce everything else that we need, from the following principles:

- If $(I - \mathbb{H})f = c$ and $f|_{\partial} = 0$ then $(I - \mathbb{H})\partial_{\alpha'} f = 0$, since $[\partial_{\alpha'}, \mathbb{H}]f = 0$ by (339).

³⁸ $\frac{1}{Z_{,\alpha}} \in C^0$ because $\frac{1}{Z_{,\alpha}} \in L^{\infty}$ and $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \in L^2$.

- If $(I - \mathbb{H})f = 0$ and $(I - \mathbb{H})g = c(t)$ then $(I - \mathbb{H})(fg) = 0$, since fg is the boundary value of the product of two periodic holomorphic functions, one of which goes to 0 as $y \rightarrow -\infty$, and therefore the product goes to 0 as $y \rightarrow -\infty$.
- If $(I - \mathbb{H})f = 0$ and $f|_{\partial} = 0$, then $(I - \mathbb{H})D_{\alpha'}f = 0$, since $(I - \mathbb{H})\partial_{\alpha'}f = 0$ and $(I - \mathbb{H})\frac{1}{Z_{,\alpha'}} = c$.
- Therefore, if $G(z, t)$ is a periodic holomorphic function on $\Omega(t)$ going to zero as $y \rightarrow -\infty$, that is,

$$(I - \mathbb{H})G(z(h^{-1}(\alpha', t), t), t) = (I - \mathbb{H})(G \circ Z) = 0, \quad (364)$$

then $(I - \mathbb{H})G_z(z(h^{-1}(\alpha', t), t), t) = 0$, since $G_z \circ Z = D_{\alpha'}(G \circ Z)$.

- If $G(z, t)$ is a periodic holomorphic function on $\Omega(t)$ going to zero as $y \rightarrow -\infty$, so $(I - \mathbb{H})(G \circ Z) = 0$, then $(I - \mathbb{H})G_t \circ Z = 0$.³⁹ $G_t \circ Z$ is periodic holomorphic, since it is the limit of periodic holomorphic functions. It remains to show that $\oint G_t \circ Z = 0$. Note that $\oint G \circ Z = 0$ for all time. Also, since $\Phi(\Phi^{-1}(\alpha', t), t) = \alpha'$, we have $(\Phi_z \circ Z) \cdot (\Phi^{-1})_t + \Phi_t \circ Z = 0$, and therefore

$$(\Phi^{-1})_t = (-Z_{,\alpha'})\Phi_t \circ Z. \quad (365)$$

Therefore, using (365) and the fact $Z = \Phi^{-1}$, we have

$$\begin{aligned} 0 &= \frac{d}{dt} \oint G(Z(\alpha', t), t) = \oint G_t(Z(\alpha', t), t) + \oint G_z \circ Z (\Phi^{-1})_t(\alpha', t) \\ &= \oint G_t \circ Z - \oint (\partial_{\alpha'}(G \circ Z))\Phi_t \circ \Phi^{-1}. \end{aligned} \quad (366)$$

It therefore suffices to show that the second integral is zero. To do this, we use $(I - \mathbb{H})\partial_{\alpha'}(G \circ Z) = 0$, integration by parts, the adjoint property (337) and (363),

$$\begin{aligned} \oint (\partial_{\alpha'}(G \circ Z))\Phi_t \circ \Phi^{-1} &= \oint \{\mathbb{P}_H(\partial_{\alpha'}(G \circ Z))\} \Phi_t \circ \Phi^{-1} \\ &= \oint (\partial_{\alpha'}(G \circ Z))\mathbb{P}_A\Phi_t \circ \Phi^{-1} = -\oint (G \circ Z)\partial_{\alpha'}(\mathbb{P}_A\Phi_t \circ \Phi^{-1}) \\ &= 0. \end{aligned} \quad (367)$$

A.3.1 Identities

We now present the various identities. These all follow from the above principles and the fact that the quantities have periodic boundary behavior. In general, they can be derived by an inductive argument, and later identities follow from the earlier ones.

$$(I - \mathbb{H})\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = 0 \quad (368)$$

$$(I - \mathbb{H})D_{\alpha'}^k \bar{Z}_t = 0, \quad \text{for } 0 \leq k \leq 3 \quad (369)$$

$$(I - \mathbb{H})\partial_{\alpha'}D_{\alpha'}^k \bar{Z}_t = 0, \quad \text{for } 0 \leq k \leq 2 \quad (370)$$

$$(I - \mathbb{H})\left\{\frac{1}{Z_{,\alpha'}}D_{\alpha'}^k \bar{Z}_t\right\} = 0, \quad \text{for } 0 \leq k \leq 2 \quad (371)$$

Let $F = \bar{v}$ be the conjugate velocity. F is a periodic holomorphic function on $\Omega(t)$ going to 0 as $y \rightarrow -\infty$, so $(I - \mathbb{H})\{F \circ Z\} = 0$. We have the following statements about F :

$$(I - \mathbb{H})\{(Z_{,\alpha'})^j D_{\alpha'}^k (F_t \circ Z)\} = 0, \quad \text{for } j = 0, 1, \quad k = 0, 1 \quad (372)$$

³⁹Note that this argument does not apply to $\Phi_t \circ Z$ itself, because Φ is not periodic.

$$(I - \mathbb{H})(Z_{,\alpha'})^j D_{\alpha'}^k (F_{tt} \circ Z) = 0, \quad \text{for } j = 0, 1, k = 0, 2. \quad (373)$$

Recall that we showed in §3.2.3 that $F_t \circ z = \bar{z}_{tt} - (D_{\alpha'} \bar{z}_t) z_t$ and $F_{tz} \circ z = D_{\alpha'} (\bar{z}_{tt} - (D_{\alpha'} \bar{z}_t) z_t)$. We begin with the basic identity

$$(I - \mathbb{H}) D_{\alpha'}^k (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) = 0, \quad \text{for } 0 \leq k \leq 3. \quad (374)$$

This gives

$$(I - \mathbb{H}) \partial_{\alpha'} D_{\alpha'}^k (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) = 0, \quad \text{for } 0 \leq k \leq 2. \quad (375)$$

We also have

$$(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) \partial_{\alpha'} D_{\alpha'}^2 (\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) \right\} = 0 \quad (376)$$

$$(I - \mathbb{H}) \partial_{\alpha'} (\Phi_t \circ Z) = 0 \quad (377)$$

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right)^k \right\} = 0, \quad \text{for } k = 1, 2 \quad (378)$$

$$(I - \mathbb{H}) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) = 0 \quad (379)$$

$$(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \mathbb{H} \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right) \right) \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right) \right\} = 0 \quad (380)$$

$$(I - \mathbb{H}) \left\{ \left(\mathbb{P}_H \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right)^2 D_{\alpha'}^2 \bar{Z}_t \right\} = 0 \quad (381)$$

$$(I - \mathbb{H}) \left\{ (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_H \frac{Z_t}{Z_{,\alpha'}} \right) (I + \mathbb{H}) D_{\alpha'} Z_t \right\} = 0 \quad (382)$$

Finally, we have one, rare, equation where the mean is not zero.⁴⁰ Using (338),

$$\begin{aligned} (I - \mathbb{H}) & \left\{ ((I + \mathbb{H}) D_{\alpha'} Z_t) \mathbb{P}_H \left[(\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right] \right\} \\ &= \frac{1}{2} \left(\oint D_{\alpha'} Z_t \right) \left(\oint (\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \left(\mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right) \right). \end{aligned} \quad (383)$$

A.3.2 Mean conditions

We have implicitly in the preceding section shown that various quantities are mean-zero, but we don't use that fact other than in those identities. We will at one point, though, require an explicit mean-zero condition, to use a variant of the Sobolev inequality. This is that

$$\oint (D_{\alpha'} \bar{Z}_t)^2 d\alpha' = 0. \quad (384)$$

We note that $\oint (D_{\alpha'} \bar{Z}_t) = 0$ because $(I - \mathbb{H}) D_{\alpha'} \bar{Z}_t = 0$. Therefore, $D_{\alpha'} \bar{Z}_t$ is the boundary value of a periodic holomorphic function going to 0 as $y \rightarrow -\infty$, so its square will also be the boundary value of a periodic holomorphic function going to 0 as $y \rightarrow -\infty$.

B Useful inequalities and identities

We present here assorted inequalities and identities that we need in our paper. None of the results here are original, so we omit the proofs in most cases. We refer the reader to [Kin14] for details of the proofs.

⁴⁰By adjointness, we can write the second mean as $\oint (\mathbb{P}_H \partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t) \frac{Z_t}{Z_{,\alpha'}} = \oint Z_t D_{\alpha'}^3 \bar{Z}_t$.

B.1 Sobolev inequalities and the Peter-Paul trick

We present here the one-dimensional Sobolev inequality we use in our proof.

Proposition 7 (Weighted Sobolev Inequality with ε). *Let $\varepsilon > 0$. Then for all $f \in C^1(-1, 1) \cap L^2(-1, 1)$,*

$$\|f\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \|f\|_{L^2(\frac{1}{\omega})} + \varepsilon \|f'\|_{L^2(\omega)} + \|f\|_{L^2} \quad (385)$$

for any weight $\omega \geq 0$.

Furthermore

$$\oint f^2 = 0 \Rightarrow \|f\|_{L^\infty} \lesssim \frac{1}{\varepsilon} \|f\|_{L^2(\frac{1}{\omega})} + \varepsilon \|f'\|_{L^2(\omega)}. \quad (386)$$

The second version, (386), will give us slightly more flexibility; the condition $\oint f^2 = 0$ will actually follow from $\oint f = 0$ for f the boundary value of a periodic holomorphic function on $[-1, 1] \times (-\infty, 0)$ (see the discussion for (384)), and so is not so constraining. We omit the proof since it is fairly standard.

We will once use the following ‘Peter-Paul’ trick.⁴¹

Proposition 8 (Peter-Paul Trick). *Let $f \in C^1(-1, 1)$. Suppose*

$$\|D_{\alpha'} f\|_{L^2} \leq c_1 + c_2 \|f\|_{L^\infty}, \quad (387)$$

where $D_{\alpha'} := \frac{1}{Z_{\alpha'}} \partial_{\alpha'}$. Suppose further that $\|f\|_{L^\infty} < \infty$. Then

$$\|f\|_{L^\infty} \lesssim (c_2 + 1)(c_2 \|f\|_{L^2(|Z_{\alpha'}|^2)} + c_1 + \|f\|_{L^2}), \quad (388)$$

where the constant implicit in \lesssim is universal.

Proof. By the weighted Sobolev inequality (385) with $\omega = \frac{1}{|Z_{\alpha'}|^2}$ and then (387),

$$\begin{aligned} \|f\|_{L^\infty} &\leq C \left(\frac{1}{\varepsilon} \|f\|_{L^2(|Z_{\alpha'}|^2)} + \varepsilon \|D_{\alpha'} f\|_{L^2} + \|f\|_{L^2} \right) \\ &\leq C \left(\frac{1}{\varepsilon} \|f\|_{L^2(|Z_{\alpha'}|^2)} + \varepsilon (c_1 + c_2 \|f\|_{L^\infty}) + \|f\|_{L^2} \right). \end{aligned} \quad (389)$$

We choose $\varepsilon = \min(\frac{1}{2Cc_2}, 1)$, so $C\varepsilon c_2 \leq \frac{1}{2}$. Subtracting $Cc_2\varepsilon \|f\|_{L^\infty}$ from both sides, we get

$$\frac{1}{2} \|f\|_{L^\infty} \leq C \left(\frac{1}{\varepsilon} \|f\|_{L^2(|Z_{\alpha'}|^2)} + \varepsilon c_1 + \|f\|_{L^2} \right), \quad (390)$$

which gives desired inequality. \square

B.2 Derivatives and complex-valued functions

Because our functions will be complex-valued, and we will often be looking at derivatives of angular and modular parts of these functions, we note here a few elementary facts about such functions.

Let $f(\alpha) = r(\alpha)e^{i\theta(\alpha)}$, where r, θ are real-valued functions. Then

$$|\partial_\alpha |f|| \leq |f'|; \quad \left| \partial_\alpha \frac{f}{|f|} \right| \leq \left| \frac{f'}{|f|} \right|. \quad (391)$$

From $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$, we have $\mathbf{a} \geq 0$ and

$$\frac{z_{tt} + i}{|z_{tt} + i|} = i \frac{z_\alpha}{|z_\alpha|}. \quad (392)$$

We will use this fact in this paper to replace the angular part of the spatial derivative z_α with the angular part of the time derivative $z_{tt} + i$.

⁴¹So-called because one takes from Peter to give to Paul.

B.3 Hardy's inequality and commutator estimates

We present here the basic estimates we will rely on for this paper. Several of these estimates control quantities of the form $[f, \mathbb{H}]g'$ by something involving f' and g ; they thus reduce the amount of regularity required on g , at the expense of further regularity on f .

For many of these estimates, we must pay close attention to the boundary conditions. Recall that $f \in C^0(S^1)$ implies that $f|_{\partial} = 0$. Many of these estimates do not hold if this periodic boundary condition is removed. To save space, we have not always explicitly cited these boundary conditions when we quote these estimates, but they are always met, by the results in §A.2.

Proposition 9 (Hardy's Inequality). *Let $f \in C^0(S^1) \cap C^1(-1, 1)$, (and so $f|_{\partial} = 0$), with $f' \in L^2$. Then there exists $C > 0$ independent of f such that for any $\alpha' \in I$,*

$$\left| \int_I \frac{(f(\alpha') - f(\beta'))^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\beta' \right| \leq C \|f'\|_{L^2}^2. \quad (393)$$

Proposition 10 ($L^2 \times L^\infty$ Estimate). *There exists a constant $C > 0$ such that for any $f \in C^0(S^1) \cap C^1(-1, 1)$ with $f' \in L^2$, $g \in C^0[-1, 1]$ with $g' \in L^p$ for some $p > 1^{42}$ (so $f|_{\partial} = 0$, though possibly $g|_{\partial} \neq 0$),*

$$\|[f, \mathbb{H}]\partial_{\alpha'} g\|_{L^2} \leq C \|f'\|_{L^2} \|g\|_{L^\infty}. \quad (394)$$

Proof. This result is the periodic modification of a result from [Wu09], which in turn is a consequence of the $T(b)$ theorem [DJS85].

We begin by integrating by parts,

$$[f, \mathbb{H}]\partial_{\alpha'} g = \mathbb{H}(f'g) - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' + \frac{1}{2i} (f(\alpha') - f(\beta')) \cot(\frac{\pi}{2}(\alpha' - \beta')) g(\beta') \Big|_{\partial}, \quad (395)$$

where we have a boundary term because we didn't place periodic boundary assumption on g . We control the first term by the L^2 boundedness of \mathbb{H} and Hölder. We control the last term by Hardy's inequality (393).

We handle the second term by using the identity

$$\left(\frac{\pi}{2}\right)^2 \frac{1}{\sin^2(\frac{\pi}{2}\alpha')} = \sum_{l \in \mathbb{Z}} \frac{1}{(\alpha' + 2l)^2} \quad (396)$$

and reducing it to the real line version, Proposition 3.2 in [Wu09]. For details, see [Kin14]. □

Proposition 11 ($L^2 \times L^\infty$ Estimate Variant). *There exists a constant $C > 0$ such that for any $f \in C^0(S^1) \cap C^1(-1, 1)$ with $f' \in L^2$, $g \in L^\infty[-1, 1]$ (and so $f|_{\partial} = 0$ but possibly $g|_{\partial} \neq 0$),*

$$\left\| \int \frac{(f(\alpha') - f(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} g(\beta') d\beta' \right\|_{L^2} \leq C \|f'\|_{L^2} \|g\|_{L^\infty}. \quad (397)$$

This is just the second term on the RHS of (395).

Proposition 12 ($L^\infty \times L^2$ Estimate [Cal65]). *There exists a constant $C > 0$ such that for any $f \in C^1[-1, 1] \cap C^0(S^1)$, $g \in C^0(S^1) \cap C^1(-1, 1)$ with $g' \in L^p$ for some $p > 1^{43}$ (and so $f|_{\partial} = g|_{\partial} = 0$),*

$$\|[f, \mathbb{H}]\partial_{\alpha'} g\|_{L^2} \leq C \|f'\|_{L^\infty} \|g\|_{L^2}. \quad (398)$$

⁴² We require this only to ensure that $[f, \mathbb{H}]g'$ is well-defined.

⁴³ We assume $g' \in L^p$ only to ensure $[f, \mathbb{H}]g'$ is well-defined.

To prove Proposition 12, we begin with the integration-by-part formula (395), noting that the third term, the boundary term, is zero since $f|_{\partial} = g|_{\partial} = 0$. We handle the first term by the L^2 boundedness of \mathbb{H} and Hölder, and the second term by identity (396), reducing it to the classical result on \mathbb{R} by [Cal65].⁴⁴

Proposition 13. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$ and $g' \in L^p$ for some $p > 1$ (and so $f|_{\partial} = g|_{\partial} = 0$),*

$$\|[f, \mathbb{H}] \partial_{\alpha'} g\|_{L^2} \leq C \|f'\|_{L^2} \|g\|_{\dot{H}^{1/2}}. \quad (399)$$

Proof. We integrate by parts, first rewriting $\partial_{\beta'} g(\beta') = \partial_{\beta'} (g(\beta') - g(\alpha'))$:

$$[f, \mathbb{H}] \partial_{\alpha'} g = \frac{1}{2i} \int f_{\beta'}(\beta') \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) (g(\beta') - g(\alpha')) d\beta' - \frac{1}{2i} \int \frac{\pi}{2} \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (g(\beta') - g(\alpha')) d\beta', \quad (400)$$

where there is no boundary term because of the periodic boundary conditions.

For the first term, we apply Cauchy-Schwarz:

$$\left| \int f'(\beta') \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) (g(\beta') - g(\alpha')) d\beta' \right| \leq \|f'\|_{L^2} \left(\int |g(\alpha') - g(\beta')|^2 \left| \cot^2\left(\frac{\pi}{2}(\alpha' - \beta')\right) \right| d\beta' \right)^{1/2}. \quad (401)$$

Taking L^2 of this in α' and using the boundedness of cosine to replace \cot^2 with $\frac{1}{\sin^2}$, we get the needed estimate.

For the second term, we use Cauchy-Schwarz:

$$\left\| \int \frac{f(\alpha') - f(\beta')}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} (g(\beta') - g(\alpha')) d\beta' \right\|_{L^2_{\alpha'}} \leq \left(\int \int \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\beta' \int \frac{|g(\alpha') - g(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\beta' d\alpha' \right)^{1/2}, \quad (402)$$

and then we use Hardy's inequality (393) to f to get our inequality (399). \square

Proposition 14. *There exists a constant $C > 0$ such that for any $f \in \dot{H}^{1/2}, g \in L^2$ (and so $f|_{\partial} = 0$),*

$$\|[f, \mathbb{H}] g\|_{L^2} \leq C \|f\|_{\dot{H}^{1/2}} \|g\|_{L^2}. \quad (403)$$

Proof. This is immediate by Cauchy-Schwarz and the boundedness of cosine. \square

Proposition 15. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f', g' \in L^2$ and $h \in L^2$ (and so $f|_{\partial}, g|_{\partial} = 0$),*

$$\|[f, g; h]\|_{L^2} := \left\| \frac{\pi}{4i} \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta' \right\|_{L^2} \leq C \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}. \quad (404)$$

Proof. By Cauchy-Schwarz,

$$\begin{aligned} & \left| \int \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta' \right| \\ & \leq \left(\int \left| \frac{f(\alpha') - f(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} \right|^2 d\beta' \right)^{1/2} \left(\int \left| \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') \right|^2 d\beta' \right)^{1/2}. \end{aligned} \quad (405)$$

Now we take the L^2 of this in the α' variable. By Hardy's inequality (393), we control the f factor by $\|f'\|_{L^2}$, and are left with

$$\left(\int \int \left| \frac{g(\alpha') - g(\beta')}{\sin(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') \right|^2 d\beta' d\alpha' \right)^{1/2}. \quad (406)$$

Applying Fubini and then using Hardy's inequality (393) once more gives the result. \square

⁴⁴The result was later extended by [CM78].

Proposition 16. *There exists a constant $C > 0$ such that for any $f \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$, $g \in L^2$ (and so $f|_{\partial} = 0$),*

$$\|[f, \mathbb{H}]g\|_{L^\infty} \leq C \|f'\|_{L^2} \|g\|_{L^2}. \quad (407)$$

Proof. Estimate (407) holds by Cauchy-Schwarz and Hardy's inequality (393). \square

Proposition 17. *There exists a constant $C > 0$ such that for any $f, g \in C^1(-1, 1) \cap C^0(S^1)$ with $f', g' \in L^2$, and $h \in L^2$ (and so $f|_{\partial}, g|_{\partial} = 0$),*

$$\|\partial_{\alpha'}[f, [g, \mathbb{H}]]h\|_{L^2} \lesssim \|f'\|_{L^2} \|g'\|_{L^2} \|h\|_{L^2}. \quad (408)$$

Proof. We differentiate:

$$\begin{aligned} \partial_{\alpha'} \operatorname{pv} \frac{1}{2i} \int (f(\alpha') - f(\beta'))(g(\alpha') - g(\beta')) \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) h(\beta') d\beta' \\ = f'[g, \mathbb{H}]h + g'[f, \mathbb{H}]h - \operatorname{pv} \frac{1}{2i} \int \frac{\pi (f(\alpha') - f(\beta'))(g(\alpha') - g(\beta'))}{\sin^2\left(\frac{\pi}{2}(\alpha' - \beta')\right)} h(\beta') d\beta'. \end{aligned} \quad (409)$$

We control the first two terms by Hölder and then (407). We control the last term by (404). \square

Proposition 18 (Higher-Order Calderon Commutator [CM78]). *There exists a constant $C > 0$ such that for any $f \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^\infty$, $h \in C^0(S^1) \cap C^1(-1, 1)$ with $h' \in L^p$ for some $p > 1$ (and so $f|_{\partial}, h|_{\partial} = 0$)*

$$\|[f, f; \partial_{\alpha'} h]\|_{L^2} \leq C \|f'\|_{L^\infty}^2 \|h\|_{L^2}. \quad (410)$$

Proof. The proof is entirely analogous to the proof of (398), and now follows from the work of [CM78], which extends the original result of [Cal65] used for (398) to allow two difference quotient factors, instead of one. To move from \mathbb{R} to our compact domain, we do the same infinite summation argument, with the only difference being that instead of (396), we use $(\frac{\pi}{2})^2 \partial_{\alpha'} \frac{1}{\sin^2(\frac{\pi}{2}\alpha')} = -\sum \frac{1}{(\alpha' + 2l)^3}$. \square

B.4 The $\dot{H}^{1/2}$ norm

We present here the following proposition.

Proposition 19. *Let $f \in C^1(-1, 1) \cap C^0(S^1)$ with $f' \in L^2$. Then*

$$(I - \mathbb{H})f = \oint f \Rightarrow \|f\|_{\dot{H}^{1/2}}^2 = \int i(\partial_{\alpha'} f) \bar{f} d\alpha'. \quad (411)$$

(411) holds because for f satisfying the assumption of Proposition 19,

$$\int i(\partial_{\alpha'} f) \bar{f} d\alpha' = \int i(\partial_{\alpha'} \mathbb{H} f) \bar{f} d\alpha', \quad (412)$$

and $i \partial_{\alpha'} \mathbb{H} f = |D|f$, where $|D|$ is the positive operator satisfying $|D|^2 = -\partial_{\alpha'}^2$. It is easy to see that $\int i(\partial_{\alpha'} f) \bar{f} d\alpha'$ is real-valued by integration by parts.

B.5 Commutator identities

We include here for reference the various commutator identities that are necessary.

$$[\partial_t, D_\alpha] = -(D_\alpha z_t) D_\alpha; \quad (413)$$

$$[\partial_t, D_\alpha^2] = [\partial_t, D_\alpha] D_\alpha + D_\alpha [\partial_t, D_\alpha] = -2(D_\alpha z_t) D_\alpha^2 - (D_\alpha^2 z_t) D_\alpha; \quad (414)$$

$$[\partial_t^2, D_\alpha] = \partial_t [\partial_t, D_\alpha] + [\partial_t, D_\alpha] \partial_t = (-D_\alpha z_{tt}) D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t) D_\alpha \partial_t. \quad (415)$$

To calculate $[i\mathbf{a}\partial_\alpha, D_\alpha]$, we use $i\mathbf{a}z_\alpha = z_{tt} + i$ (17) to rewrite $i\mathbf{a}\partial_\alpha = i\mathbf{a}z_\alpha D_\alpha = (z_{tt} + i)D_\alpha$. Therefore

$$[i\mathbf{a}\partial_\alpha, D_\alpha] = [(z_{tt} + i)D_\alpha, D_\alpha] = -(D_\alpha z_{tt})D_\alpha. \quad (416)$$

Adding (415) and (416), we conclude that

$$[\partial_t^2 + i\mathbf{a}\partial_\alpha, D_\alpha] = (-2D_\alpha z_{tt})D_\alpha + 2(D_\alpha z_t)^2 D_\alpha - 2(D_\alpha z_t)D_\alpha \partial_t. \quad (417)$$

Because $[(\partial_t^2 + i\mathbf{a}\partial_\alpha), D_\alpha^2] = [(\partial_t^2 + i\mathbf{a}\partial_\alpha), D_\alpha]D_\alpha + D_\alpha[(\partial_t^2 + i\mathbf{a}\partial_\alpha), D_\alpha]$, we have

$$\begin{aligned} [(\partial_t^2 + i\mathbf{a}\partial_\alpha), D_\alpha^2] &= (-4D_\alpha z_{tt})D_\alpha^2 + 4(D_\alpha z_t)^2 D_\alpha^2 - 2(D_\alpha z_t)D_\alpha \partial_t D_\alpha - (2D_\alpha^2 z_{tt})D_\alpha \\ &\quad + 4(D_\alpha z_t)(D_\alpha^2 z_t)D_\alpha - 2(D_\alpha^2 z_t)D_\alpha \partial_t - 2(D_\alpha z_t)D_\alpha^2 \partial_t. \end{aligned} \quad (418)$$

C Summary of notation

We list here the various notations we've introduced in the paper. See also §1.3 for a discussion of the conventions used.

- $f|_\partial := f(1, t) - f(-1, t)$.
- ν is the angle the water wave makes with the wall. See Figure 2.
- $I := [-1, 1]$ (except when it's used for the identity or as an abbreviation for a quantity to be controlled).
- $\Re z, \Im z$ are the real and imaginary parts of a complex number z .
- Function spaces $C^k(-1, 1), C^k[-1, 1], C^k(S^1), L^p$, and $\dot{H}^{1/2}$ are defined in §1.3. We define $\|f\|_{\dot{H}^{1/2}} := \left(\frac{\pi}{8} \iint \frac{|f(\alpha') - f(\beta')|^2}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} d\alpha' d\beta' \right)^{1/2}$.
- $\oint f = \oint_I f := \frac{1}{2} \int_I f(\beta') d\beta'$.
- $z(\alpha, t)$ is the Lagrangian parametrization, and $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity. $z_{tt} = \partial_t z_t$, etc.
- $\mathbf{a} = \frac{|\bar{z}_{tt} - i|}{|z_\alpha|} = \frac{-\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|}$. We refer to $-\frac{\partial P}{\partial \mathbf{n}}$ as the Taylor coefficient; \mathbf{n} is the outward-facing normal to $\Sigma(t)$.
- $h : \alpha \mapsto \alpha'$ is defined by $h(\alpha) = \Phi(z(\alpha, t), t)$ and gives the Riemannian coordinates, where Φ is the Riemann mapping defined in §3.1. $\partial_{\alpha'}(f \circ h^{-1}) = \frac{\partial_\alpha f}{h_\alpha} \circ h^{-1}$. $d\alpha' = h_\alpha d\alpha$. Full details are in §3.1.
- α and β are our variables in Lagrangian coordinates; α' and β' are our variables in Riemannian coordinates.
- \mathbb{H} is the Hilbert transform in Riemannian coordinates, defined by

$$\mathbb{H}f(\alpha') := \frac{1}{2i} \text{pv} \int_I \cot\left(\frac{\pi}{2}(\alpha' - \beta')\right) f(\beta') d\beta'. \quad (419)$$

We use \mathfrak{H} to refer to the Hilbert transform for the curved domain in Lagrangian coordinates.

- We define $\mathbb{P}_A := \frac{(I - \mathbb{H})}{2}$ and $\mathbb{P}_H := \frac{(I + \mathbb{H})}{2}$ as the antiholomorphic and holomorphic projections.
- $[f, g; h](\alpha') := \frac{\pi}{4i} \int \frac{(f(\alpha') - f(\beta'))(g(\alpha') - g(\beta'))}{\sin^2(\frac{\pi}{2}(\alpha' - \beta'))} h(\beta') d\beta'$, the higher-order Calderon commutator.
- We use $F(z(\alpha, t), t) := \bar{z}_t(\alpha, t)$ at several points (and do not to use F for any other purpose).
- $Z := z \circ h^{-1}, Z_t := z_t \circ h^{-1}, Z_{tt} := z_{tt} \circ h^{-1}$. $Z_{,\alpha'} = \partial_{\alpha'}(z \circ h^{-1}), Z_{t,\alpha'} = \partial_{\alpha'}(z_t \circ h^{-1})$, etc.

- Compositions and inverses are always with respect to the spatial variable.
- $\mathcal{A} := (\mathbf{a}h_\alpha) \circ h^{-1}$, $\mathcal{A}_t := (\mathbf{a}_t h_\alpha) \circ h^{-1}$.
- $D_\alpha := \frac{1}{z_\alpha} \partial_\alpha$, $|D_\alpha| := \frac{1}{|z_\alpha|} \partial_\alpha$; $D_{\alpha'} := \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}$, $|D_{\alpha'}| := \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}$.
- $A_1 = \mathcal{A}|Z_{,\alpha'}|^2 = iZ_{,\alpha'}(\bar{Z}_{tt} - i) \in \mathbb{R}$ (45). On changing variables, we have

$$A_1 \circ h = \frac{\mathbf{a}|z_\alpha|^2}{h_\alpha}, \quad (420)$$

originally derived at (50); we use this repeatedly without citation. We often use $A_1 \geq 1$ (47), and also use $\frac{1}{Z_{,\alpha'}} = i \frac{\bar{Z}_{tt} - i}{A_1}$ (49).

- We define our energies in §4.1. We define generic energies $E_{a,\theta}$ and $E_{b,\theta}$, and then specialize to $E_a := E_{a,D_\alpha^2 \bar{z}_t}$ and $E_b := E_{b,D_\alpha \bar{z}_t}$. We use G_θ to describe the RHS of the equation $(\partial_t^2 + i\mathbf{a}\partial_\alpha)\theta = G_\theta$. For $\theta = D_\alpha^k \bar{z}_t$, $G_\theta = D_\alpha^k(-i\mathbf{a}_t \bar{z}_\alpha) + [\partial_t^2 + i\mathbf{a}\partial_\alpha, D_\alpha^k] \bar{z}_t$.
- $\Theta := \theta \circ h^{-1}$; $B := \left(\frac{h_{t\alpha}}{h_\alpha} - \Re D_{\alpha'} \bar{z}_t\right) \circ h^{-1}$; $\psi := \left(\frac{h_\alpha \theta}{z_\alpha}\right) \circ h^{-1}$ (165).
- See §1.3 for a discussion of how broadly I, II, I_1, I_{12} , etc., are defined. In short, they are unambiguous within each section, but ambiguous between sections.
- We use $C(E)$ to represent a polynomial of the energy E .

D Main quantities controlled

We list here the various quantities that are controlled by our energy, for ease of reference. We don't list every single quantity we have controlled, but we do include any quantities that we give at the end of a concluding inequality without further explanation.

- In §5.1, we controlled

$$\begin{aligned} & \|D_{\alpha'}^2 \bar{Z}_{tt}\|_{L^2}, \|D_{\alpha'}^2 Z_{tt}\|_{L^2}, \|D_{\alpha'}^2 \bar{Z}_t\|_{L^2}, \|D_{\alpha'}^2 Z_t\|_{L^2}, \|D_\alpha \partial_t D_\alpha \bar{z}_t\|_{L^2(h_\alpha d\alpha)}, \\ & \left\| \frac{1}{Z_{,\alpha'}} D_{\alpha'}^2 \bar{Z}_t \right\|_{\dot{H}^{1/2}}, \|D_{\alpha'} \bar{Z}_{tt}\|_{L^\infty}, \|D_{\alpha'} Z_{tt}\|_{L^\infty}, \|D_{\alpha'} \bar{Z}_t\|_{L^\infty}, \|D_{\alpha'} Z_t\|_{L^\infty}, \\ & \|\bar{Z}_{tt,\alpha'}\|_{L^2}, \|\bar{Z}_{t,\alpha'}\|_{L^2}, \int |D_\alpha \bar{z}_t|^2 \frac{d\alpha}{\mathbf{a}}, \int |D_\alpha \bar{z}_{tt}|^2 \frac{d\alpha}{\mathbf{a}}, \left\| \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}, \|Z_{tt} + i\|_{L^\infty}, \|A_1\|_{L^\infty}. \end{aligned} \quad (421)$$

- $\left\| \frac{\mathbf{a}_t}{\mathbf{a}} \right\|_{L^\infty} = \left\| \frac{\mathcal{A}_t}{\mathcal{A}} \right\|_{L^\infty}$ is controlled at (114) in §5.2.
- $\left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}$ is controlled at (121) in §5.3.
- $\left\| \frac{h_{t\alpha}}{h_\alpha} \right\|_{L^\infty}$ is controlled at (129) in §5.4.
- $\|(I + \mathbb{H})D_{\alpha'} Z_t\|_{L^\infty}$ is controlled at (143) in §5.7.
- $\left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ is controlled at (141) in §5.6. The related term $\left\| (Z_{tt} + i)\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^\infty}$ is also estimated there.
- $\left\| \partial_{\alpha'} \mathbb{P}_A \frac{Z_t}{Z_{,\alpha'}} \right\|_{L^\infty}$ is controlled at (147) in §5.8. The related term $\left\| \mathbb{P}_A \left(Z_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|_{L^\infty}$ we estimated at (146).

References

- [ABZ14a] Thomas Alazard, Nicolas Burq, and Claude Zuily. Cauchy theory for the gravity water waves system with non localized initial data, 2014, arXiv:1305.0457 [math.AP].
- [ABZ14b] Thomas Alazard, Nicolas Burq, and Claude Zuily. Strichartz estimates and the Cauchy problem for the gravity water waves equations. 2014, arXiv:1404.4276 [math.AP].
- [ABZar] Thomas Alazard, Nicolas Burq, and Claude Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, To Appear.
- [AD13] Thomas Alazard and Jean-Marc Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves, 2013, arXiv:1305.4090 [math.AP].
- [AM05] David M. Ambrose and Nader Masmoudi. The zero surface tension limit of two-dimensional water waves. *Comm. Pure Appl. Math.*, 58(10):1287–1315, 2005.
- [Cal65] A.-P. Calderón. Commutators of singular integral operators. *Proc. Nat. Acad. Sci. U.S.A.*, 53:1092–1099, 1965.
- [CL00] Demetrios Christodoulou and Hans Lindblad. On the motion of the free surface of a liquid. *Comm. Pure Appl. Math.*, 53(12):1536–1602, 2000.
- [CM78] Ronald R. Coifman and Yves Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque*. Société Mathématique de France, Paris, 1978. With an English summary.
- [Cra85] Walter Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10(8):787–1003, 1985.
- [CS07] Daniel Coutand and Steve Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930, 2007.
- [DJS85] G. David, J.-L. Journé, and S. Semmes. Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana*, 1(4):1–56, 1985.
- [Ebi87] David G. Ebin. The equations of motion of a perfect fluid with free boundary are not well posed. *Comm. Partial Differential Equations*, 12(10):1175–1201, 1987.
- [GMS12] P. Germain, N. Masmoudi, and J. Shatah. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)*, 175(2):691–754, 2012.
- [HIT14] John Hunter, Mihaela Ifrim, and Daniel Tataru. Two dimensional water waves in holomorphic coordinates, 2014, arXiv:1401.1252 [math.AP].
- [IP13] Alexandru D. Ionescu and Fabio Pusateri. Global solutions for the gravity water waves system in 2d, 2013, arXiv:1303.5357 [math.AP].
- [Jou83] Jean-Lin Journé. *Calderon-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderon*, volume 994 of *Lecture Notes in Math.* Springer, 1983.
- [Kin14] Rafe Hand Kinsey. *A Priori Estimates for Two-Dimensional Water Waves with Angled Crests*. Ph.D. thesis, University of Michigan, Ann Arbor, 2014.
- [Lan05] David Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654 (electronic), 2005.
- [LC25] T. Levi-Civita. Détermination rigoureuse des ondes permanentes d’amplitude finie. *Math. Ann.*, 93(1):264–314, 1925.

- [Lin05] Hans Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)*, 162(1):109–194, 2005.
- [Nal74] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104–210, 254, 1974.
- [Sto47] G. G. Stokes. On the theory of oscillatory waves. *Trans. Cambridge Philos. Soc.*, 8:441–455, 1847.
- [Tay50] Geoffrey I. Taylor. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. *Proc. Roy. Soc. London. Ser. A.*, 201:192–196, 1950.
- [Wu97] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.
- [Wu99] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.
- [Wu09] Sijue Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177(1):45–135, 2009.
- [Wu11] Sijue Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184(1):125–220, 2011.
- [Wu12] Sijue Wu. On a class of self-similar 2d surface water waves, 2012, arXiv:1206.2208 [math.AP].
- [Yos82] Hideaki Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18(1):49–96, 1982.

R.H. Kinsey, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109
E-mail address: rkinsey@umich.edu

S. Wu (Corresponding author), DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109
E-mail address: sijue@umich.edu